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# ON ORDERING PERISHABLE INVENTORY UNDER ERLANG DEMAND*

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**ABSTRACT**

Existing models for describing optimal ordering policies for perishable inventory cast the problem as a multidimensional dynamic program, the dimensionality being one less than the product lifetime in periods. An approach developed in previous work takes explicit account of outdating in the single period model. Formulas for the expected quantity of any new order which will outdate are developed for the case where the demand has a stationary Erlang distribution. A modified version of the one period model is shown to yield a reasonable approximation to the stationary optimal policy.

## INTRODUCTION

An inventory problem which has received very little attention in the literature until recently is that of describing optimal ordering policies for a commodity with a fixed lifetime. A significant portion of the inventory problems encountered are of this type. One example is the problem of blood bank management; in most areas, the shelf life of whole blood is limited to 21 days. Many drugs must be discarded after a fixed period as well. The food industry is another which is characterized by fixed life commodities.

There has been some effort directed towards analyzing ordering policies for a commodity which decays continuously in time. Ghare and Schrader [4] and Emmons [2] have considered such a problem. The earliest analysis of the fixed life case was undertaken by Van Zyl [11] who restricted consideration to a product with a two period lifetime and considered only ordering and shortage costs. The model was of a periodic review type and utilized the functional method of dynamic programming. More recently, Nahmias and Pierskalla [6] considered a more general model of the two period lifetime problem. Extensions of these two models to the case of product with an arbitrary lifetime has been considered by Fries [3] and Nahmias [8], respectively.

## I. THE ONE PERIOD MODEL

We consider the case of a single product at a single installation. The model we consider is a periodic review model; that is, ordering takes place at the start of a period and costs are incurred during the period. The period length is arbitrary, but fixed.

We will make the following assumptions: (1) All orders are placed at the start of the period and received instantaneously. (2) All stock arrives new. (3) Demands in successive periods are independent identically distributed random variables with known distribution $F$ and density $f$. (4) Inventory is de-

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pleted according to a FIFO policy; that is, oldest first. (5) Costs are charged linearly against (a) ordering, (b) holding, (c) unsatisfied demand (runouts), and (d) deterioration (outdates). (6) If the product has not been depleted by demand by the time it reaches age $m$ periods (that is, after $m$ demands have been realized), then it deteriorates and must be discarded at a specified per-unit cost. (7) Unsatisfied demand is backlogged.

The first problem to consider in developing a model is describing the state variable of the system. Define $x_i$ = amount of product on hand which will outdate (perish) exactly $i$ periods into the future so that the vector $\mathbf{x}=(x_{m-1}, x_{m-2}, \ldots, x_1)$ represents the total initial inventory at each age level. The decision variable, $y$, represents the amount of new product being ordered.

To motivate our approach consider the expected single period cost function. Letting $c$, $h$, $r$, $\theta$ (all positive) be the respective unit cost for ordering, holding, shortage and outdating, $f$ the demand density, the expected cost of ordering $y$ when $x$ is on hand is:

$$\hat{L}(\mathbf{x}, y) = cy + h\int_0^{x+y}(x+y-t)f(t)dt + r\int_{x+y}^{\infty}(t-x-y)f(t)dt + \theta\int_0^{x_1}(x_1-t)f(t)dt,$$

where for convenience $x=\sum_{i=1}^{m-1} x_i$.

If we define $\mathbf{s}(y, \mathbf{x}, t)$ = the vector of inventories on hand one period later if $\mathbf{x}$ is presently on hand, $y$ is the quantity of new product ordered and $t$ is the realization of demand in the period (that is, the transfer function) then

$$\mathbf{s}(y, \mathbf{x}, t) = (s_{m-1}(y, \mathbf{x}, t), \ldots, s_1(y, \mathbf{x}, t))$$

where

$$s_i(y, \mathbf{x}, t) = (x_{i+1} - (t - \sum_{j=1}^{i} x_j)^+)^+, \qquad 1 \leq i \leq m-2$$

$$s_{m-1}(y, \mathbf{x}, t) = \begin{cases} y-(t-x)^+ & \text{if demand is backlogged} \\ (y-(t-x)^+)^+ & \text{if sales are lost,} \end{cases}$$

and $g^+ = \max(g, 0)$.

Fries [3] proceeds to analyze the functional equations

$$C_n(\mathbf{x}) = \inf_{y \geq 0}\left\{\hat{L}(\mathbf{x}, y) + \alpha\int_0^{\infty} C_{n-1}(\mathbf{s}(y, \mathbf{x}, t))f(t)dt\right\}.$$

There is one problem with this approach. Minimizing $\hat{L}(\mathbf{x}, y)$ with choice of $y$ yields the same policy one would obtain for a nonperishable problem since the final term is independent of $y$. If $c=h=0$ then a minimizing value will not even exist. In fact one must perform the recursive computations

for $m$ periods before the effects of outdating can be seen on each component of the optimal ordering function. Due to the multidimensional state variable such computations may not be feasible.

It was primarily for this reason that Nahmias and Pierskalla [6, 7] and Nahmias [8] adopted a different approach. The actual quantity of new product which will outdate in $m$ periods is a random variable which depends on the vector of inventories on hand as well as the transfer function and the demands over the next $m$ periods. If we can determine the expected value of that random variable, a cost of outdating could be included in the single period cost function which would be a function of the present order, $y$.

A crucial assumption here is that the depletion policy is FIFO (that is, oldest first). Unless the oldest inventory is used to satisfy demand, the expected outdating of the present order may depend upon future decisions. This is not a very strong assumption as Nahmias [9] and Pierskalla and Roach [10] have demonstrated the optimality of FIFO as a depletion policy for perishable inventory under far more general circumstances.

Fixing the vector of on hand inventories, $\mathbf{x}$, and letting $D_1, D_2, \ldots, D_m$ represent demands in the next $m$ periods, define the sequence of random variables $B_j$ by the rule:

$$B_j = \begin{cases} 0 & \text{for } j=0 \\ (D_j + B_{j-1} - x_j)^+ & \text{for } 1 \leq j \leq m-1. \end{cases}$$

In [8], it is demonstrated that $R^* = (y - D_m - B_{m-1})^+$ is the amount of $y$ which perishes. In order to determine the distribution of $R^*$ it is convenient to define:

$$G_n(\mathbf{x}(n)) = P\{D_n + B_{n-1} \leq x_n\},$$

where $\mathbf{x}(n) = (x_n, \ldots, x_1)$. Although we have suppressed the notation, it should be recognized that $B_j$ depends on both $\mathbf{x}(j)$ and $(D_j, \ldots, D_1)$. It follows that

$$G_1(x_1) = P\{D_1 \leq x_1\} = F(x_1).$$

As is demonstrated in the appendix

$$G_n(\mathbf{x}(n)) = \int_0^{x_n} G_{n-1}(v + x_{n-1}, \mathbf{x}(n-2)) f(x_n - v)\, dv \qquad \text{for } n \geq 2,$$

from which one obtains

$$E(R^*) = \int_0^y G_m(u, \mathbf{x})\, du.$$

At this point we can define our expected one period cost function

$$L(\mathbf{x}, y) = cy + h \int_0^{x+y} (x + y - t) f(t)\, dt + r \int_{x+y}^{\infty} (t - x - y) f(t)\, dt + \theta \int_0^y G_m(u, \mathbf{x})\, du.$$

Notice that the final term depends on $y$ as well as the vector $\mathbf{x}$. Nahmias [8] proceeds to analyze the functional equations

$$C_n(\mathbf{x}) = \inf_{y \geq 0} \left\{ L(\mathbf{x}, y) + \alpha \int_0^\infty C_{n-1}(\mathbf{s}(y, \mathbf{x}, t) f(t) dt \right\}.$$

This model will also yield reasonable results when only costs associated with shortage and outdating are present ([7]), but still suffers from the fact that the computation of an optimal policy requires the solution of a multidimensional dynamic program. However, the first period decision now includes in an explicit way the effects of outdating and thus should provide a better approximation to the stationary optimal policy than the previous approach. The computation of the one period policy is confounded by the fact that the function $G_m(u; \mathbf{x})$ is defined by a sequence of convolutions whose computation can be costly in both time and accuracy if numerical integration techniques are used. In the next section we will show how explicit representations of this function can be obtained when the demand distribution is either exponential or Erlang.

## II. EXPONENTIAL AND ERLANG DEMAND

For convenience assume that all costs are linear. We will make the assumption that inventory remaining at the end of the final period is salvaged at a return of $c$ per unit and backlogged inventory can be made up by an emergency order at a cost $c$ per unit. This causes the ordering region to remain stationary ([8]) and would tend to eliminate the property that an optimal policy orders fewer units as one approaches the end of the horizon ([1]). Since $L(\mathbf{x}, y)$ is convex in $y$ for all $\mathbf{x}$, with the inclusion of the salvage value assumption, it follows that the one period decision, $y_1(\mathbf{x})$ uniquely solves

$$\frac{\partial L(\mathbf{x}, y_1(\mathbf{x}))}{\partial y} - \alpha c = 0$$

or $c(1-\alpha) + (h+r)F(x+y_1(\mathbf{x})) - r + \theta G_m(y_1(\mathbf{x}), \mathbf{x}) = 0$. Assume that

$$f(t) = \begin{cases} \lambda \exp(-\lambda t) & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

Define the sequence of $n$th order posynomials $P_n(\mathbf{x}(n))$ by the recursion

$$P_n(\mathbf{x}(n)) = \begin{cases} 1 & \text{if } n = 0 \\ \displaystyle\int_0^{x_n} P_{n-1}(v + x_{n-1}, \mathbf{x}(n-2)) dv & \text{if } 1 \leq n \leq m. \end{cases}$$

We then have the following:

THEOREM 1: Assume the demand has the exponential distribution with parameter $\lambda$ in every period. Then

$$G_n(\mathbf{x}(n)) = G_{n-1}(x_n, \ldots, x_2) - \lambda^{n-1} \cdot \exp(-\lambda w_n) P_{n-1}(x_n, \ldots, x_2)$$

where $w_n = \sum_{i=1}^{n} x_i$.

PROOF: Since $G_0(t) = 1$, $P_0 = 1$ and $G_1(x_1) = F(x_1)$ the result follows for $n = 1$. Assume it holds for $1, 2, \ldots, n-1$. Then

$$G_n(\mathbf{x}(n)) = \int_0^{x_n} G_{n-1}(v + x_{n-1}, x_{n-2}, \ldots, x_1) f(x_n - v)\, dv$$

$$= \int_0^{x_n} [G_{n-2}(v + x_{n-1}, x_{n-2}, \ldots, x_2) - \lambda^{n-2} \exp(-\lambda(v + w_{n-1})) \cdot P_{n-2}(v + x_{n-1}, \ldots, x_2)] \lambda \exp(-\lambda(x_n - v))\, dv$$

$$= \int_0^{x_n} G_{n-2}(v + x_{n-1}, x_{n-2}, \ldots, x_2) f(x_n - v)\, dv - \lambda^{n-1} \exp(-\lambda w_n) \int_0^{x_n} P_{n-2}(v + x_{n-1}, \ldots, x_2)\, du$$

$$= G_{n-1}(x_n, \ldots, x_2) - \lambda^{n-1} \exp(-\lambda w_n) P_{n-1}(x_n, \ldots, x_2). \qquad \square$$

Hence an explicit expression for $G_m(y, \mathbf{x})$ can be easily obtained when demand is exponential. For example, for $m = 4$ we have

$$P_0 = 1$$

$$P_1(x_1) = x_1$$

$$P_2(x_2, x_1) = x_2^2/2 + x_1 x_2$$

$$P_3(x_3, x_2, x_1) = x_3^3/6 + \{(x_3^2/2) \cdot (x_1 + x_2)\} + \{(x_3/2) \cdot (x_2^2 + 2x_1 x_2)\}$$

and

$$G_4(y, x_3, x_2, x_1) = 1 - \exp(-\lambda y) - \lambda \exp(-\lambda(y + x_3)) \cdot P_1(y) - \lambda^2 \exp(-\lambda(y + x_3 + x_2)) \cdot P_2(y, x_3) - \lambda^3 \exp(-\lambda(y + x_3 + x_2 + x_1)) \cdot P_3(y, x_3, x_2).$$

Similar, although more complicated expressions, may be obtained when the demand distribution is Erlang with parameters $\lambda$ and $r$. In that case:

$$f(t) = \begin{cases} 0 & \text{for } t < 0, \\ (r-1)!^{-1} \lambda^r t^{r-1} \exp(-\lambda t) & \text{for } t \geq 0. \end{cases}$$

Define for $0 \leq k \leq r$ the polynomials

$$P_k(x_1) = x_1^k/k!, \qquad 0 \leq k \leqq r.$$

Suppose that $1 \leq p \leq n$. Then we will determine the collection of $k$th order posynomials $P_k(x_p, \ldots, x_1)$ whenever $r(p-1) \leq k \leq rp$ by induction on $p$. Suppose we have generated $P_k$ for $0 \leq k \leq rp$. Since for each value of $r(p-1) \leq k \leq rp$, $P_k(x_p, \ldots, x_1)$ is a $k$th order posynomial in the $p$ variables $x_p, \ldots, x_1$, it follows that we may write

$$P_k(v+x_p, \ldots, x_1) = \sum_{j=0}^{k} a_{j,k} v^j,$$

where the coefficients $a_{j,k}$ are of the form $b_0 x_1^{b_1} x_2^{b_2} \ldots x_p^{b_p}$ for some constants $b_0, b_1, \ldots, b_p$. We now define $P_{k+r}$ by the rule

$$P_{k+r}(x_{p+1}, \ldots, x_1) = \sum_{j=0}^{k} a_{j,k} j! [(j+r)!]^{-1} x_{p+1}^{r+j}.$$

Continuing in this fashion we may generate the entire sequence $P_1, P_2, \ldots, P_{rn-1}$. Once these posynomials have been obtained, the functions $G_n(\mathbf{x}(n))$ may be computed recursively by:

THEOREM 2: Assume the demand is stationary with the Erlang distribution with parameters $\lambda$ and $r$. Then

$$G_n(\mathbf{x}(n)) = G_{n-1}(x_n, \ldots, x_2) - \exp(-\lambda w_n) \sum_{k=1}^{r} \lambda^{rn-k} P_{rn-k}(\mathbf{x}(n)).$$

PROOF: As in the exponential case, we obtain $G_1(x_1) = F_1(x_1)$. Assuming the result holds for $1, 2, \ldots, n-1$, then

$$G_n(\mathbf{x}(n)) = \int_0^{x_n} G_{n-1}(v+x_{n-1}, \mathbf{x}(n-2)) f(x_n - v)\, dv$$

$$= \int_0^{x_n} (G_{n-2}(v+x_{n-1}, \ldots, x_2) - \exp(-\lambda(v+w_{n-1}))$$

$$\left( \sum_{k=1}^{r} \lambda^{r(n-1)-k} P_{r(n-1)-k}(v+x_{n-1}, \ldots, x_2) \right) f(x_n - v)\, dv$$

$$= G_{n-1}(x_n, \ldots, x_2) - \exp(-\lambda w_n) \sum_{k=1}^{r} \lambda^{rn-k} ((r-1)!)^{-1}$$

$$\int_0^{x_n} P_{r(n-1)-k}(v+x_{n-1}, \ldots, x_2)(x_n - v)^{r-1} dv.$$

Suppose that

$$P_{r(n-1)-k}(v+x_{n-1}, \ldots, x_2)=\sum_{j=0}^{k} a_{j,k}v^j.$$

Then by successively integrating by parts, we obtain

$$\int_0^{x_n}\left(\sum_{j=0}^{k} a_{j,k}v^j\right)(x_n-v)^{r-1}dv=\sum_{j=0}^{k} a_{j,k}\int_0^{x_n} v^j(x_n-v)^{r-1}dv=\sum_{j=0}^{k} a_{j,k}j!(r-1)!((j+r)!)^{-1}x_n^{r+j}. \quad \square$$

It should be noted that the term $P_{r(n-1)}(\mathbf{x}(n))$ is independent of $x_1$ and this term was included in the sum for convenience only. A more exact expression for the final term would be

$$\sum_{k=1}^{r-1}\lambda^{rn-k}P_{rn-k}(\mathbf{x}(n))+\lambda^{r(n-1)}P_{r(n-1)}(x_n, \ldots, x_2).$$

As in the exponential case, it is the collection of posynomials $P_k$ which determine $G_m$. As an example let $m=3$, $r=2$, then:

$$P_j(x_1)=x_1^j/j!, \qquad 0\leqslant j\leqslant 2$$

$$P_3(x_2, x_1)=x_2^3/3!+x_1x_2^2/2!$$

$$P_4(x_2, x_1)=(2!)^{-1}((x_1x_2)^2/2!+2x_1x_3^3/3!+2!x_2^4/4!)$$

$$P_5(x_3, x_2, x_1)=(3!)^{-1}(3!x_3^5/5!+3\cdot 2!x_3^4x_2/4!+3\cdot x_3^3x_2^2/3!+x_2^3x_3^2/2!)$$
$$+(2!)^{-1}(2!x_1x_3^4/4!+2x_1x_2x_3^3/3!+x_1x_2^2x_3^2/2!)$$

$$G_3(y, x_2, x_1)=1-e^{-\lambda y}-\lambda ye^{-\lambda y}-(\lambda^2/2!)y^2e^{-\lambda(y+x_2)}-\lambda^3P_3(y, x_2)e^{-\lambda(y+x_2)}-\lambda^4P_4(y, x_2)e^{-\lambda(y+x_1+x_2)}$$
$$-\lambda^5P_5(y, x_2, x_1)e^{-\lambda(y+x_1+x_2)}.$$

## III. COMPARISON WITH STATIONARY POLICY

Once the form of $G_m$ has been determined, computing the one period optimal policy, $y_1(\mathbf{x})$, can be accomplished with relative efficiency since $L(\mathbf{x}, y)$ is convex in $y$. Normally one would choose a grid in $R^{m-1}$ and determine $y_1(\mathbf{x})$ at each $\mathbf{x}$ within the grid. The precision obtained would correspond to the fineness of the chosen grid.

The key question is under what circumstances, if any, can the stationary optimal policy $y^*(\mathbf{x})$ be approximated by $y_1(\mathbf{x})$. We first note that the expected holding and shortage costs are functions of only $x+y$, the total inventory on hand after ordering. If only these costs are present and it is optimal to order to some point in any period, it must be optimal to order to the same point in every other period. Hence the terms involving holding and shortage costs will not affect the stationarity of the ordering policy. However, both costs associated with ordering and outdating depend on both $x$ and $y$ (and not merely the sum $x+y$) and will cause the optimal policy to be nonstationary.

Note that if $\theta=0$, the solution of the one period model yields $F(x+y_1(\mathbf{x}))=(r-c(1-\alpha))/(h+r)$ which implies $x+y_1(\mathbf{x})=\bar{x}$, the stationary critical number for the nonperishable problem. It is proven in [8] that $x+y_1(\mathbf{x})<\bar{x}$, and in this case our method yields no additional information about the stationary policy.

Hence we require initially that $\theta>0$. In order to eliminate the nonstationarity caused by the ordering cost term we will let $c=0$. Hence $y_1(\mathbf{x})$ solves

$$(h+r)F(x+y_1(\mathbf{x}))-r+\theta G_m(y_1(\mathbf{x}),x)=0.$$

In all such cases tested, it was found that $y_1(\mathbf{x})<y^*(\mathbf{x})$; however, it was also found that $|y_1(\mathbf{x})-y^*(\mathbf{x})|<|\bar{x}-x-y^*(\mathbf{x})|$ for all $x$. That is, the one period approximation was significantly better than an approximation obtained from solving the nonperishable problem. In Figure 1 $y_1(\mathbf{x})$ and $y^*(\mathbf{x})$ are compared; for this case $c=0$, $r=5$, $h=1$, $\theta=10$ and exponential demand with mean 10 was assumed.

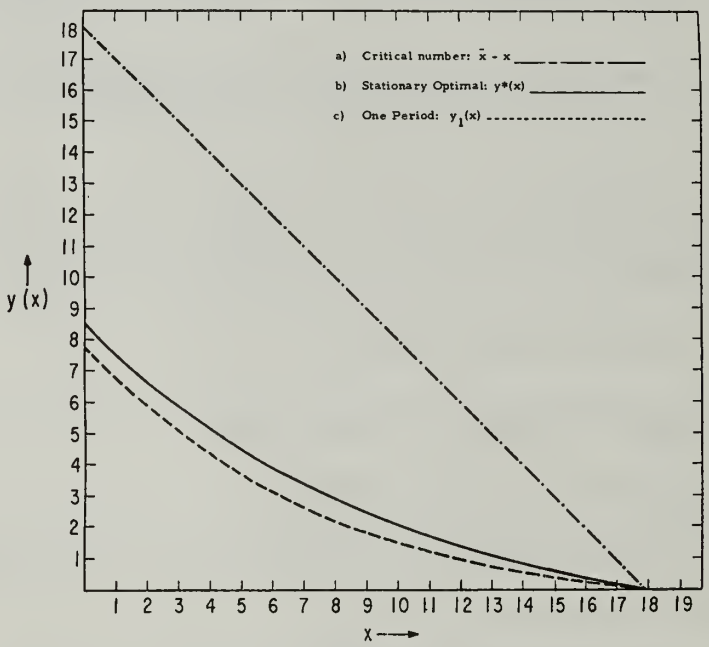


FIGURE 1. *Comparison of three policies.*

It would appear that our method is essentially without value when $\theta=0$ or $c$ is large compared to $\theta$. However, note that the term $c(1-\alpha)$ corresponds to the cost of ordering the quantity demanded the previous period, which is precisely the amount ordered when the inventory is nonperishable and one follows a critical number policy. For perishable inventory, following a critical number policy would require ordering the previous periods demand plus the quantity of inventory which perished the previous period. Since the term $\int_0^{y_1(x)} G_m(u,\mathbf{x})du$ estimates the number of units which outdate, we may approximate the additional ordering cost due to outdating by the term $\alpha c\int_0^{y_1(x)} G_m(u,\mathbf{x})du$.

The $\alpha$ is included to account for the fact that this additional cost is incurred from period two onward. Hence, $y_1(\mathbf{x})$ now solves

$$c(1-\alpha)+(h+r)f(x+y_1(\mathbf{x}))-r+(\theta+\alpha c)G_m(y_1(\mathbf{x}),\mathbf{x})=0.$$

In all cases examined, $y_1(\mathbf{x})$ computed in this manner undershot the stationary policy $y^*(\mathbf{x})$. However, the magnitude of the difference observed was not large when compared with the policy obtained by assuming the inventory was nonperishable. In Table 1 we consider a variety of cost configurations and Erlang distributions. In all cases it was assumed that $m=2$ so that the effects of outdating would be strongest. (Note that $\mathbf{x}$ has only one component so we will write $x$.) For each case we computed the following quantities:

1. max 1: the maximum deviation of $y_1(x)$ from $y^*(x)$:

$$\max_{0\leq x\leq \bar{x}} 1=\max\,(y^*(x)-y_1(x)):$$

2. mean 1: the average deviation of $y_1(x)$ from $y^*(x)$:

$$\text{mean } 1=(1/[\bar{x}])\sum_{x=0}^{[\bar{x}]}(y^*(x)-y_1(x))\,[[\bar{x}] \text{ is the largest integer } \leq \bar{x}];$$

3. max 2: the maximum deviation of $y^*(x)$ from the critical number policy:

$$\max 2=\max_{0\leq x\leq \bar{x}}(\bar{x}-x-y^*(x));$$

4. mean 2: the average deviation of $y^*(x)$ from the critical number policy:

$$\text{mean } 2=(1/[\bar{x}])\cdot\sum_{x=0}^{[\bar{x}]}(\bar{x}-x-y^*(x)).$$

TABLE 1. *Comparison of policies*

| Cost Case | Max 1 | Mean 1 | Max 2 | Mean 2 |
|---|---|---|---|---|
| 0/5/10/1 | 0.78 | 0.55 | 9.52 | 6.17 |
| 0/10/5/1 | 1.54 | 1.19 | 9.36 | 6.13 |
| 5/10/5/1 | 1.27 | 0.93 | 10.27 | 6.63 |
| 10/10/1/1 | 1.27 | 0.96 | 8.68 | 5.68 |
| 20/40/40/1 | 1.30 | 0.77 | 19.67 | 12.00 |
| 5/10/12/1 | 1.02 | 0.69 | 12.32 | 7.83 |
| 3/10/10/2 | 0.72 | 0.57 | 7.10 | 4.38 |
| 5/10/12/2 | 0.65 | 0.53 | 7.36 | 4.65 |
| 3/6/2/2 | 0.62 | 0.48 | 3.85 | 2.37 |
| 3/10/10/3 | 0.51 | 0.40 | 4.93 | 3.04 |
| 5/5/5/3 | 0.32 | 0.24 | 3.09 | 1.85 |

Each case considered can be identified by the sequence $c/r/\theta/k$. The holding cost was fixed at $h=1.0$, the discount factor $\alpha=0.95$ and mean demand at 10. Hence a cost case 0/5/10/1 would correspond to an ordering cost of zero, shortage cost of 5, and outdating cost of 10 and exponential demand

with parameter $\lambda = 0.10$. We see immediately that the critical number policy provides a poor approximation in almost every case. It overshot the optimal policy by a significant amount which varied dramatically with different cost configurations. However, $y_1(x)$ undershot the optimal policy by an amount which remained relatively stable and, in most cases, did not exceed a single unit. In cases where computing time may be expensive, our method provides an approximation which can be obtained efficiently and which does not require restricting one's attention to policies of a specific class whose shape might be radically different from the optimal.

## ACKNOWLEDGMENT

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## APPENDIX

THEOREM:

$$G_n(\mathbf{x}(n)) = \int_0^{x_n} G_{n-1}(v + x_{n-1}, \mathbf{x}(n-2)) f(x_n - v)\, dv.$$

for $1 \leq n \leq m$, and $G_0 = 1$.

PROOF: Since $G_1(x_1) = P\{D_1 \leq x_1\} = F(x_1)$ and

$$G_1(x_1) = \int_0^{x_1} f(x_1 - v)\, dv = F(x_1),$$

the result holds true for $n = 1$.

Assume that the theorem is true for 1, 2, . . ., $n-1$ (where $n-1 < m$). Then

$$G_n(\mathbf{x}(n)) = P\{D_n + B_{n-1} \leq x_n\} = P\{D_n + (D_{n-1} + B_{n-2} - x_{n-1})^+ \leq x_n\}$$

$$= P\{D_n \leq x_n, D_{n-1} + B_{n-2} - x_{n-1} \leq 0\} + P\{D_n + D_{n-1} + B_{n-2} - x_{n-1} \leq x_n, D_{n-1} + B_{n-2} - x_{n-1} \geq 0\},$$

which results from conditioning on the exhaustive and mutually exclusive events $\{D_n + B_{n-2} - x_n \leq 0\}$ and $\{D_n + B_{n-2} - x_n \geq 0\}$. Considering each term separately,

$$P\{D_n \leq x_n, D_{n-1} + B_{n-2} - x_{n-1} \leq 0\} = F(x_n) G_{n-1}(\mathbf{x}(n-1))$$

by the inductive assumption and the fact that $D_n$ and $D_{n-1} + B_{n-2}$ are statistically independent.

For convenience let $Y_n = D_{n-1} + B_{n-2}$. By induction $Y_n$, has distribution function $G_{n-1}$, so that the second term above becomes

$$P\{D_n + Y_n \leq x_n + x_{n-1}, Y_n \geq x_{n-1}\} = \int_{x_{n-1}}^{x_{n-1}+x_n} dG_{n-1}(u, \mathbf{x}(n-2)) F(x_n + x_{n-1} - u)$$

$$= \int_{x_{n-1}}^{x_{n-1}+x_n} G_{n-1}(u, \mathbf{x}(n-2)) f(x_n + x_{n-1} - u)\, du - F(x_n) G_{n-1}(\mathbf{x}(n-1)),$$

by an integration by parts. By combining both expressions and performing the change of variable $v=u-x_{n-1}$ we obtain

$$G_n(\mathbf{x}(n))=\int_0^{x_n} G_{n-1}(v+x_{n-1},\mathbf{x}(n-2))f(x_n-v)dv,$$

as required.

□

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# INSTANT TRANSPORTATION SOLUTIONS

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**ABSTRACT**

This paper presents direct noniterative methods for solving such known problems as shoploading and aggregate scheduling. There is given a simple optimal rule for the shoploading problem which is quite surprising. The crucial point in solving this problem is what not to assign rather than what to assign. The development of those methods was possible since the discussed problems can be converted into a special transportation model where the given cost coefficients $c_{ij}$ assume a form $c_{ij}=u_i+v_j$.

## 1. INTRODUCTION

Consider the following problems:

Minimize

$$z=\sum_{i=1}^{m}\sum_{j=1}^{n}(u_i+v_j)x_{ij}, \tag{1}$$

subject to

$$\left.\begin{array}{l}\displaystyle\sum_{j=1}^{n}x_{ij}\begin{matrix}\leqslant\\=\\\geqslant\end{matrix}a_i\\[2ex]\displaystyle\sum_{i=1}^{m}x_{ij}\begin{matrix}\leqslant\\=\\\geqslant\end{matrix}b_i\end{array}\right\}\quad 1\leqslant i\leqslant m,\qquad 1\leqslant j\leqslant n \tag{2}$$

$$x_{ij}\geqslant 0\qquad \text{for all } 1\leqslant i\leqslant m,\qquad 1\leqslant j\leqslant n, \tag{3}$$

where $u_i$, $v_j$, and $b_j$ are given ($a_i>0$, $b_j>0$).

Denote for convenience

$$a=\sum_i a_i,\quad b=\sum_j b_j.$$

Note that each of those nine problems is a special transportation type model with $c_{ij}=u_i+v_j$.

There are presented direct (noniterative) solution procedures for the stated problem (see Appendix 1) for this problem.

To demonstrate that (1), (2), and (3) has practical applications we show that such known problems as shoploading ([4] and [6]), and aggregate scheduling ([1], [2], [4], [5], and [6]) can be presented in a (1), (2), and (3) form.

Write (2) as equations by adding if necessary slack variables $x_i$ and $y_j$. Then (2) becomes

$$(4)\qquad \left.\begin{aligned} &\sum_{j=1}^{n} x_{ij}+\begin{pmatrix}1\\0\\-1\end{pmatrix}x_i=a_i \\ &\sum_{i=1}^{m} x_{ij}+\begin{pmatrix}1\\0\\-1\end{pmatrix}y_j=b_j \\ &x_{ij}\geqslant 0, x_i\geqslant 0, y_j\geqslant 0 \end{aligned}\right\} 1\leqslant i\leqslant m, \qquad 1\leqslant j\leqslant n.$$

The main idea of the solution method is to consider an equivalent transportation problem

$$(5)\qquad \min z'=\sum_{i=1}^{m}\begin{pmatrix}1\\0\\-1\end{pmatrix}(-u_i)x_i+\sum_{j=1}^{n}\begin{pmatrix}1\\0\\-1\end{pmatrix}(-v_j)y_j,$$

subject to (4).*

Problem (4) and (5) can be solved as follows:

Find $x_i$ and $y_j$ minimizing (5) subject to

$$\sum_{i=1}^{m}\begin{pmatrix}1\\0\\-1\end{pmatrix}x_i+\sum_{j=1}^{n}\begin{pmatrix}1\\0\\-1\end{pmatrix}y_j=\mathrm{a}-\mathrm{b}$$

and (if the coefficient at $x_i$ and $y_j$ is $+1$)

$$x_i\leqslant a_i, \qquad y_j\leqslant b_j.$$

This linear programming problem with one constraint and upper bounds on some variables is solvable instantaneously. Replace in (4) the optimal $x_i$ and $y_j$. Any set of $x_{ij}$ satisfying (4) is optimal.

Note that if (2) are strict equations then any feasible solution of (2) is optimal. The goal of this note is to call attention to problem (1), (2), and (3), its structure, and its properties. It is possible to write *explicit formulas* for the optimal slack variables. One should stress that once $x_i$ and $y_j$ are found the decision which feasible assignment $x_{ij}$ to choose becomes immaterial since *every* feasible assignment is optimal.

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*Numbers $1, 0, -1$ correspond to signs $\leqslant, =, \geqslant$ in (2), respectively.

Let us label each case of (1), (2), and (3) as follows:
Case $(\leqslant, \geqslant)$ stands for

$$\min z = \sum_i \sum_j (u_i + v_j)x_{ij}, \text{ subject to } \sum_j x_{ij} \leqslant a_i, \sum_i x_{ij} \geqslant b_j.$$

The appendix presents a solution to cases $(=,=)$, $(=, \leqslant)$, $(\geqslant,=)$, and $(\geqslant, \geqslant)$. We leave for the reader the solution of the remaining cases.

## 2. SOME PROPERTIES OF PROBLEM (1), (2), AND (3)

Assume

(6) $$c_{ij} = u_i + v_j \geqslant 0 \qquad \text{for all } 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n.$$

It has been shown (Appendix 1) that

$$\text{case } (\geqslant, \geqslant) \langle = \rangle [\text{case } (\geqslant,=) \text{ or case } (=, \geqslant) \text{ or case } (=,=)].$$

This equivalence may not be true for the case when

$$c_{ij} \neq u_i + v_j \text{ (see [3] and [7])}.$$

One can also show that (6) implies:

$$\text{case } (\leqslant, \geqslant) \langle = \rangle \text{ case } (\leqslant,=), \qquad \text{case } (\geqslant, \leqslant) \langle = \rangle \text{ case } (=, \leqslant).$$

REMARK 1: Let $c_{ij} = u_i + v_j \leqslant 0$ for all $i$ and $j$. Then

$$\text{case } (\leqslant, \leqslant) \langle = \rangle \text{ case } [(\leqslant,=) \text{ or case } (=, \leqslant) \text{ or case } (=,=)].*$$

## 3. THE SHOPLOADING PROBLEM

The Shoploading Problem (see [6], pp. 279–287) is defined as follows. There are $m$ facilities which can handle either one of $n$ different jobs. The problem is how to assign those facilities to jobs so as to maximize the total profit under such limitations as available hours for each facility and maximal demand (in numbers of pieces) for each job.

The following notations are utilized:

$\bar{a}_i$ – available hours
$w_i$ – wage rate
} for facility $i = 1, \ldots, m$.

$\bar{b}_j$ – demand (number of pieces)
$r_j$ – material cost per piece
$s_j$ – selling price per piece
} for job $j = 1, \ldots, n$.

*Depending on whether $a > b$, $a < b$, or $a = b$.

$p_{ij}$ — production rate (number of pieces per hour) of facility $i$ for job $j$.
$y_{ij}$ — number of pieces of job $j$ processed on facility $i$.
$\bar{c}_{ij}$ — profit per piece of job $j$ processed on facility $i$.

Then

$$\bar{c}_{ij}=s_j-(r_j+w_i/p_{ij}).$$

Assume also that the production rates have the following property:

$$p_{ij}=p_j q_i \tag{7}$$

where $p_j$ and $q_i$ are positive constants.

REMARK 2: Without loss of generality we can assume that $q_i=1$ for some $i$, say $t$. Then facility $t$ will be considered as a standard machine with $p_i$ as standard production rates.

Consider the following transformations:

$$\left.\begin{aligned} a_i&=\bar{a}_i q_i \\ b_j&=\bar{b}_j/p_j \\ x_{ij}&=y_{ij}/p_j \\ \bar{\bar{c}}_{ij}&=\bar{c}_{ij}p_j=[s_j-(r_j+w_i/p_j q_i)]p_j \end{aligned}\right\} 1\leqslant i\leqslant m, \qquad 1\leqslant j\leqslant n. \tag{8}$$

Hence

$$\bar{\bar{c}}_{ij}=-w_i/q_i+(s_j-r_j)p_j.$$

Observe that the $a_i$, $b_j$ and $x_{ij}$* are expressed in standard hours.

Finally to convert it into a minimization problem, define

$$c_{ij}=-\bar{\bar{c}}_{ij}=w_i/q_i-(s_j-r_j)p_j, \qquad 1\leqslant i\leqslant m, \qquad 1\leqslant j\leqslant n. \tag{9}$$

Formulas (8) and (9) transform the shoploading problem into case $(\leqslant, \leqslant)$ with

$$u_i=w_i/q_i, \qquad v_j=-.(s_j-r_j)p_j. \tag{10}$$

Note that $c_{ij}\leqslant 0$ (since $\bar{\bar{c}}_{ij}\geqslant 0$).

Hence (Remark 1) case $(\leqslant, \leqslant)$ $\langle=\rangle$ case $(\leqslant, =)$ or case $(=, \leqslant)$ or case $(=, =)$ depending on whether $a>b$, $a<b$, or $a=b$.

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*Notice that $\bar{c}_{ij}y_{ij}=\bar{\bar{c}}_{ij}x_{ij}$.

By applying the procedure presented in the appendix, one can solve each of those three cases.

Denote the optimal nonslack variables by $\bar{x}_{ij}$. Then $\bar{y}_{ij}=p_j\bar{x}_{ij}$ are optimal decision variables of the original problem.

EXAMPLE: Consider the example from [6], pp. 279–287. The data are summarized in the following table.

| $i, j=$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\bar{a}_i$ | 36 | 54 | 80 | 33⅓ | |
| $w_i$ | 6 | 4.2 | 4.8 | 3.6 | |
| $\bar{b}_j$ | 300 | 210 | 240 | 1800 | 400 |
| $r_j$ | 0.5 | 0.6 | 0.8 | 1 | 1.2 |
| $s_j$ | 4 | 4 | 5 | 3 | 3 |
| $p_j$ | 6 | 7 | 6 | 30 | 25 |
| $q_i$ | 0.5 | 1 | 0.8 | 0.6 | |

Then from (8) we get the following results:

| $i, j=$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $a_i$ | 18 | 54 | 64 | 20 | |
| $b_j$ | 50 | 30 | 40 | 60 | 16 |
| $w_i/q_i$ | 12 | 4.2 | 6 | 6 | |
| $(s_j-r_j)p_j$ | 21 | 23.8 | 25.2 | 60 | 45 |

Since in our example $a=156<b=196$ we have a $(=, \leqslant)$ case.

Instead of using matrix $C=\{c_{ij}\}$*

$$C=\begin{bmatrix} -9 & -11.8 & -13.2 & -48 & -33 \\ -16.8 & -19.6 & -21 & -55.8 & -40.8 \\ -15 & -17.8 & -19.2 & -54 & -39 \\ -15 & -17.8 & -19.2 & -54 & -39 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

*See page 284 in [6] where the author actually solved a maximization problem working with the profit matrix, i.e. matrix $\bar{\bar{C}}=-C$.

as the author of [6] did we solve an equivalent $(=, \leqslant)$ case by replacing $C$ by $C' = \{c_{ij} + \alpha_i + \beta_j\}$ with $\alpha_i = -w_i/q_i$, $1 \leqslant i \leqslant 4$, $\alpha_5 = 0$ and $\beta_j = (s_j - r_j)p_j$, $1 \leqslant j \leqslant 5$. Then

$C' =$

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 18 |
| 0 | 0 | 0 | 0 | 0 | 54 |
| 0 | 0 | 0 | 0 | 0 | 64 |
| 0 | 0 | 0 | 0 | 0 | 20 |
| 21 | 23.8 | 25.2 | 60 | 45 | $40 = 196 - 156$ |
| 50 | 30 | 40 | 60 | 16 | |

The $a_i$ and $b_j$ appear on the right hand side and below $C'$.

According to the solution procedure for case $(=, \leqslant)$ (which amounts to the minimum row method for the last row of $C'$) we have $y_1 = \min(50, 40) = 40$. The remaining $y_j = 0$.

Find any feasible solution for the updated rim conditions:

| | | | | | |
|---|---|---|---|---|---|
| | | | | | 18 |
| | | | | | 54 |
| | | | | | 64 |
| | | | | | 20 |
| 40 | | | | | ~~40~~ |
| ~~50~~ 10 | 30 | 40 | 60 | 16 | |

One such solution found by applying the northwest corner rule is given below:

| | | | | |
|---|---|---|---|---|
| 10 | 8 | | | |
| | 22 | 32 | | |
| | | 8 | 56 | |
| | | | 4 | 16 |
| 40 | | | | |

The optimal solution of the original $4 \times 5$ shoploading problem (where $y_{ij} = p_j x_{ij}$) is then:

| facilities \ jobs | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 60 | 56 | | | |
| 2 | | 154 | 192 | | |
| 3 | | | 48 | 1680 | |
| 4 | | | | 120 | 400 |
| | 60 | 210 | 240 | 1800 | 400 |

As seen the only job with unsatisfied demand is job 1 (60 pieces instead of 300).

## 4. THE AGGREGATE SCHEDULING PROBLEM

Bowman [2] showed that a problem of minimizing total costs of regular and overtime production (under capacity restrictions) as well as inventory storage while meeting sales requirements can be presented as a transportation problem. The corresponding cost matrix and other relevant data for a three period sales case are given below.

| | Sales periods 1 | 2 | 3 | Final invent. | Slack | Maximal supply |
|---|---|---|---|---|---|---|
| Initial inventory | 0 | $c$ | $2c$ | $3c$ | 0 | $a_1$ |
| Regular 1 | $r$ | $r+c$ | $r+2c$ | $r+3c$ | 0 | $a_2$ |
| Overtime 1 | $w$ | $w+c$ | $w+2c$ | $w+3c$ | 0 | $a_3$ |
| Regular 2 | $\infty$ | $r$ | $r+c$ | $r+2c$ | 0 | $a_4$ |
| Overtime 2 | $\infty$ | $w$ | $w+c$ | $w+2c$ | 0 | $a_5$ |
| Regular 3 | $\infty$ | $\infty$ | $r$ | $r+c$ | 0 | $a_6$ |
| Overtime 3 | $\infty$ | $\infty$ | $w$ | $w+c$ | 0 | $a_7$ |
| Demand | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | |

The following notations are utilized

$c$—carrying unit cost per period

$r$—unit production cost

$w$—overtime unit production cost

$T$—number of sales periods.

It is easy to see that outside the $\infty$ area and the $T+2-th$ column

$$c_{ij}=u_i+v_j,$$

where

$$\left.\begin{array}{l} u_1=0 \\ u_{2s}=r-(s-1)c \\ u_{2s+1}=w-(s-1)c \\ v_j=(j-1)c \end{array}\right\} \quad 1\leqslant s\leqslant T, \qquad 1\leqslant j\leqslant T+1. \tag{11}$$

Hence this problem is a $(\leqslant,=)$ case, with additional constraints:

$$\sum_{i=1}^{3}(a_i-x_i)-b_1\geqslant 0,\ \sum_{i=1}^{5}(a_i-x_i)-\sum_{j=1}^{2}b_j\geqslant 0,\ \ldots,\ \sum_{i=1}^{2T+1}(a_i-x_i)-\sum_{j=1}^{T-1}b_j\geqslant 0 \tag{12}$$

and can be solved by the method for this case. However, in view of (12) the formulas for $x_i$ are much more complex than (17) (see Appendix 1) for the regular $(\leqslant,=)$ case.

It is well known ([4], p. 444) that the minimum column method (provided we handle the columns in the order 1, 2, . . ., $T+2$) will produce an optimal solution.

Appendix 2 shows how to find the optimal $x_i$ directly.

## APPENDIX 1

## SOLUTION OF PROBLEM (1), (2), AND (3)

### 1. PRELIMINARY REMARKS

Let us recall the following well known fact. Consider two transportation problems with identical equality constraints, and with $a=b$.

PROPERTY 1: If the transportation cost matrices are $\{c_{ij}\}$ and $\{c'_{ij}\}=\{c_{ij}+\alpha_1+\beta_j\}$ ($\alpha_i$ and $\beta_j$ are arbitrary constraints) respectively then both problems are equivalent. The idea of the solution procedure is to solve the equivalent transportation problem (4)–(5) (see section 1) with the cost matrix $\{c'_{ij}\}$ and where $\alpha_i$ and $\beta_j$ $1\leqslant i\leqslant m+1, 1\leqslant j\leqslant n+1$ satisfy the following conditions

$$\left.\begin{array}{ll} \alpha_i=-u_i & 1\leqslant i\leqslant m \\ \beta_j=-v_j & 1\leqslant j\leqslant n \end{array}\right\} \tag{13}$$

$$\alpha_{m+1}=0 \tag{14}$$

(15) $$\beta_{n+1}=0.$$

Assume also

$$u_{i_1} \geq u_{i_2} \geq \ldots \geq u_{i_m}$$

(16)

$$v_{j_1} \geq v_{j_2} \geq \ldots \geq v_{j_n}$$

## 2. CASE (=, =)

Here $C'=\{c'_{ij}\}$ is a $m \times n$ matrix. Since $\alpha_i$ and $\beta_j$ satisfy (13), $c'_{ij}=0$ for all $1 \leq i \leq m$ $1 \leq j \leq n$. Then (4) and (5) becomes

$$\min z' = \sum_{i=1}^{m} \sum_{j=1}^{n} 0 \cdot x_{ij},$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i$$

$$\sum_{i=1}^{m} x_{ij} = b_j$$

$$x_{ij} \geq 0.$$

CONCLUSION: Any feasible solution of (=, =) is optimal.

## 3. CASE (≤, =)*

Define $\alpha_i$, $\beta_j$ by (13) and (15). The $m \times (n+1)$ matrix $C'$ is then

$$c'_{ij}=\begin{cases}0 & 1 \leq i \leq m,\ 1 \leq j \leq n \\ -u_i & 1 \leq i \leq m,\ j=n+1\end{cases}$$

and (4)–(5) assumes the following form

(5) $$\min : z' = \sum_{i=1}^{m} (-u_i)x_i,$$

*Note that $a \geq b$.

subject to

$$\left.\begin{array}{l} \sum_{j=1}^{n} x_{ij} + x_i = a_i \\ \sum_{i=1}^{m} x_{ij} = b_j \\ \sum_{i=1}^{m} x_i = a - b \overset{\text{df}}{=} b_{n+1} \\ x_i \geqslant 0 \qquad x_{ij} \geqslant 0 \end{array}\right\} \qquad 1 \leqslant i \leqslant m, \qquad 1 \leqslant j \leqslant n. \tag{4}$$

To solve this problem find first the $x_i$ which minimize (5) subject to

$$\sum_{i=1}^{m} x_i = a - b$$

$$x_i \leqslant a_i \qquad 1 \leqslant i \leqslant m.$$

It is easy to see that in view of (16)† the optimal $x_i$ are

$$\left.\begin{array}{l} x_{i_k} = \min\left(a_{i_k},\ b_{n+1} - \sum_{t=1}^{k-1} x_{i_t}\right), \qquad k = 1, \ldots, m \\ \text{where} \\ b_{n+1} - \sum_{t=1}^{k-1} x_{i_t} \geqslant 0 \end{array}\right\} \tag{17}$$

To find the remaining optimal $x_{ij}$ proceed as follows. Insert the optimal values $x_i$ into (4). Then any solution $x_{ij}$ which satisfies (4) is optimal.

## 4. Case (=, ≥)

Define $\alpha_i$ and $\beta_j$ by (13) and (14). The $(m+1) \times n$ matrix $C'$ becomes

$$c'_{ij} = \begin{cases} 0 & 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n \\ -v_j & i = m+1,\ 1 \leqslant j \leqslant n \end{cases}$$

† $-u_{i_1} \leqslant -u_{i_2} \leqslant \ldots$.

The corresponding problem (4)–(5) is then

$$\min : z' = \sum_{j=1}^{n} v_j y_j, \tag{5}$$

subject to

$$\left.\begin{array}{l} \sum_{j=1}^{n} x_{ij} = a_i \\ \sum_{i=1}^{m} x_{ij} - y_j = b_j \\ \sum_{j=1}^{n} y_j = a - b \\ x_{ij} \geq 0, \qquad y_j \geq 0 \end{array}\right\} \quad 1 \leq i \leq m, \qquad 1 \leq j \leq n. \tag{4}$$

First find $y_j$ minimizing (5)

subject to

$$\sum_{j=1}^{n} y_j = a - b$$
$$y_j \geq 0.$$

The optimal $y_j$ are (in view of (16))

$$y_{j_n} = a - b \qquad y_{j_k} = 0, \qquad k \neq n.$$

The remaining $x_{ij}$ we find in the same way as for the previous case.

**5. Case ($\geq$, $\geq$)**

$C'$ is then a $(m+1) \times (n+1)$ matrix will $\alpha_i$ and $\beta_j$ defined by (13), (14), and (15). Hence

$$c'_{ij} = \begin{cases} 0 & 1 \leq i \leq m, \ 1 \leq i \leq n \\ -u_i & 1 \leq i \leq m, \ j = n+1 \\ -v_j & i = m+1, \ 1 \leq j \leq n. \end{cases}$$

Problem (4)–(5) assumes the following form

$$\min z' = \sum_{i=1}^{m} u_i x_i + \sum_{j=1}^{n} v_j y_j \tag{5}$$

subject to

$$(4)\qquad \left.\begin{array}{ll} \sum_{j=1}^{n} x_{ij} - x_i = a_i & \\ \sum_{i=1}^{m} x_{ij} - y_j = b_j & 1 \leqslant i \leqslant m, \quad 1 \leqslant i \leqslant n. \\ x_{ij} \geqslant 0, \quad x_i \geqslant 0, \quad y_j \geqslant 0 & \end{array}\right.$$

We find first the slack variables $x_i \geqslant 0$ and $y_j \geqslant 0$ which minimize (5) subject to

$$\sum_{i=1}^{m} x_i - \sum_{j=1}^{n} y_j = b - a.$$

It is easy to see that in view of (16) all $x_i$ and $y_j$ are zero except possibly of $x_{i_m}$ and $y_{j_n}$. To find those two variables consider two cases.

(a) $u_{i_m} + v_{j_n} = \min_{i,j} (u_i + v_j) \geqslant 0$. Then

$$\begin{array}{lll} x_{i_m} = b - a, & y_{j_n} = 0 & \text{if } a \leqslant b \\ x_{i_m} = 0, & y_{j_n} = b - a & \text{if } a > b. \end{array}$$

(b) $u_{i_m} + v_{j_n} < 0$. Then

$$\begin{array}{lll} x_{i_m} = b - a + M, & y_{j_n} = M & \text{if } a \leqslant b \\ x_{i_m} = M, & y_{j_n} = a - b + M & \text{if } a > b, \end{array}$$

where $M$ is a sufficiently large positive number. Note that $\min z' = -\infty$

COROLLARY: If $c_{ij} = u_i + v_j \geqslant 0$ for all $i, j$ then

$$\text{case } (\geqslant, \geqslant) \ \langle = \rangle \ \text{case}[(\geqslant, =) \text{ or case } (=, \geqslant) \text{ or case } (=, =)]$$

depending on whether $a < b$, $a > b$, or $a = b$.

We leave for the reader the solution of the remaining five cases.

## APPENDIX 2

## THE OPTIMAL SLACKS IN THE AGGREGATE SCHEDULING PROBLEM

We have shown in section 4 that the Aggregate Scheduling Problem is a $(\leqslant, =)$ case. To find the $x_i \geqslant 0$ we are to solve the following problem.

$$\min z' = \sum_{i=1}^{2T+1} (-u_i) x_i$$

subject to

$$\sum_{i=1}^{2T+1} x_i = \sum_{i=1}^{2T+1} a_i - \sum_{j=1}^{T+1} b_j \overset{df}{=} b_{T+2}$$

and (12) assuming the $u_i$ are defined by (11).

To find the optimal $x_i$ apply a minimum column method to the $T+2-th$ column of matrix $C'=\{c'_{ij}\}$, where

$$\left.\begin{aligned} c'_{1T+2} &= 0 \\ c'_{2s,T+2} &= -r+(s-1)c \\ c'_{2s+1,T+2} &= -w+(s-1)c \end{aligned}\right\} 1 \leqslant s \leqslant T$$

in such a manner that the successively determined $x_i$ do not violate (12). Let me illustrate this procedure on a 3-period example where $C'$ the $a_i$ and $b_j$ are as follows:

| | | | | | |
|---|---|---|---|---|---|
| 0 | | | | 0 | 5 |
| | | | | −9 | 4 |
| | | | | −12 | 2 |
| ∞ | 0 | | | −7 | 2 |
| | | | | −10 | 3 |
| ∞ | | 0 | | −5 | 8 |
| | | | | −8 | 4 |
| 6 | 6 | 2 | 3 | 11 | |

Note that $b_5=11=\sum_i a_i-\sum_j b_j$, $r=9$, $w=12$, $c=2$.

Arrange the $c'_{i,T+2}$ in a nondecreasing order. The rows $i$ form the following sequence:

$$3, 5, 2, 7, 4, 6, 1.$$

Applying the minimum column method to the last column while respecting condition (12) we find

$$x_3=\min\,[2,\ 11,\ (5+4+2)-6,\ (5+4+2+2+3)-(6+6)]=\min\,(2,\ 11,\ 5,\ 4)=2$$

Update the $a_{i_k}$ and $b_{T+2}$ ($a_3=0$, $b_5=9$) and continue the procedure until the updated $b_5$ becomes zero. We have

$x_5=\min\,[3, 9, (5+4+2+3)-(6+6)]=\min\,(3+9+2)=2,$

$x_2=\min\,(4, 7, (5+4)-6, (5+4+2+1)-(6+6)]=\min\,[4, 7, 3, 0]=0,$

$x_7=\min\,(4, 7)=4,$

$x_4=\min\,[2, 3, (5+4+2+1)-(6+6)]=\min\,(2, 3, 0)=0,$

and finally

$x_6=\min\,(3, 8)=3$. Since the updated $b_5$ is zero all remaining $x_i=0$.

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# A COMPUTATION PROCEDURE FOR THE EXACT SOLUTION OF LOCATION-ALLOCATION PROBLEMS WITH RECTANGULAR DISTANCES*

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**ABSTRACT**

A method is presented to locate and allocate $p$ new facilities in relation to $n$ existing facilities. Each of the $n$ existing facilities has a requirement flow which must be supplied by the new facilities. Rectangular distances are assumed to exist between all facilities. The algorithm proceeds in two stages. In the first stage a set of all possible optimal new facility locations is determined by a set reduction algorithm. The resultant problem is shown to be equivalent to finding the $p$-median of a weighted connected graph. In the second stage the optimal locations and allocations are obtained by using a technique for solving the $p$-median problem.

A set of test problems was solved; exact solutions and computation times are given.

## 1. INTRODUCTION

There frequently arises in problems of location a general structure characterized by

(1) a set of $n$ points $a_j$, $j=1, \ldots, n$, in the plane called existing facilities,

(2) known positive weights $w_j$, $j=1, \ldots, n$, representing flow requirements at each existing facility point, and

(3) a set of $p$ unlocated variable facilities.

The problem is to determine:

(1) the optimal location for each of the $p$ variable facilities, and

(2) the optimal allocation of existing facility requirements to the variable facilities so that all requirements are satisfied.

Let the $p$ unknown variable facility locations be given by $\chi_i, i=1, \ldots, p$. When the criterion to be satisfied is the minimization of average weighted distance, the problem may be stated as,

$$\text{minimize } R(\chi, w) = \sum_{i=1}^{p} \sum_{j=1}^{n} w_{ij}\, d(\chi_i, a_j)\,,$$

subject to

$$\sum_{i=1}^{p} w_{ij} = w_j, \qquad j=1, \ldots, n, \tag{1.1}$$

$$w_{ij} \geq 0, \qquad i=1, \ldots, p; \qquad j=1, \ldots, n,$$

where $\chi = (\chi_1, \ldots, \chi_p)$, $w = (w_{11}, w_{12}, \ldots, w_{1p}, w_{21}, \ldots, w_{pn})$ and $d(\chi_i, a_j)$ is the distance between variable point $\chi_i$ and fixed point $a_j$.

Problem (1.1) has a nonlinear objective function which is neither concave nor convex, in general, and may contain several local optima. The problem considered in this paper is the case where the $a_j$ are points in two-dimensional space and the $d(\chi_i, a_j)$ are rectilinear. The results may be generalized to three-dimensional space.

Problems of this type arise in many practical situations; a typical one is the location of $p$ warehouses to serve $n$ customers, where customer $j$ is located at $a_j$. Customer $j$ has a requirement of $w_j$ per unit time period. The total distribution cost in the system is proportional to the sum of weighted distances. The problem is to locate each of the $p$ uncapacitated warehouses and to simultaneously assign each customer to a warehouse in order to minimize total transportation cost.

The most common solution method for this problem involves choosing a starting location for each warehouse, allocating each customer to the nearest warehouse and then computing new optimal warehouse locations [2, 5, 23]. The cycle of allocating and locating is repeated until no further reduction in cost can be achieved. Using this heuristic solution method, however, achieves only a local minimum which may be a very poor solution. Studies by Eilon, Watson-Gandy, and Christofides [5] show that for $p$=5 and $n$=50, by using 200 trials, 61 local optima were found and the worst solution deviated from the best by 40.9 percent. Heuristic and branch-and-bound solution procedures are given by Kuenne and Soland where computational times are given for near-optimal solutions [15]. When the values of the $w_{ij}$ are known, solution methods for location problems having Euclidean distances are discussed by Love and Kraemer [16]. Network location solution methods are given by Goldman [7, 8, 9] and Goldman and Witzgall [10]. A discussion of methods of location-allocation on a graph and a branch-and-bound solution method is given by Järvinen et al. [13], a second branch-and-bound procedure is given by El-Shaieb [6] and a linear programming method is given by Revelle and Swain [22].

For problems defined with rectangular distances, near-optimal computational results are given by Kuenne and Soland. An exact solution procedure is given by Morris [19], but it is limited to relatively small problems. For the rectangular case, when the values of the $w_{ij}$ are known, multifacility location methods are given by Wesolowsky and Love [25, 26, 27], Cabot, Francis and Stary [1] and Morris [21]. For multifacility location problems incorporating generalized $l_p$ distances, a nonlinear programming solution method is given by Love and Morris [18].

In a recent article, Wendell and Hurter have shown that in location problems incorporating rectangular distances, it is only necessary to consider the "intersection points" in the convex hull as candidates for the optimal locations [24]. In this paper we present a stronger result which proves that a subset of the points in the convex hull need only be considered. Additionally, the development in Wendell and Hurter is geometrical; in the present paper an efficient computational algorithm is developed for determining this smaller set. We then show how this reduced set can be utilized to solve rectangular distances location-allocation problems with a backtrack programming procedure. Exact comparisons are difficult, but the method appears to be capable of computing exact solutions to larger problems than other methods which have appeared.

## 2. AN ALGORITHM FOR REDUCING THE CANDIDATE SOLUTION SET

Essential to the method to be outlined here is a theorem which demonstrates a property of sets of positive numbers.

THEOREM 1: Let $n$ positive numbers be given by $w_j$, $j=1, \ldots, n$.

Let $J$ be the index set $\{1, \ldots, n\}$. Let $J_r$ be a set consisting of $r$ elements of $J$, and $J_t$ be a set consisting of $t$ elements of $J$. Let the empty set be given by $\phi$. Then the two inequalities

$$\sum_{j \in J_r} w_j > \sum_{j \in (J - J_r)} w_j$$

and

$$\sum_{j \in J_t} w_j > \sum_{j \in (J - J_t)} w_j$$

cannot hold simultaneously when $J_r \cap J_t = \phi$.

PROOF OF THEOREM 1: The two inequalities may be written as

$$\sum_{j \in J_r} w_j > \sum_{j \in J_t} w_j + \sum_{j \in [J - (J_r \cup J_t)]} w_j$$

and

$$\sum_{j \in J_t} w_j > \sum_{j \in J_r} w_j + \sum_{j \in [J - (J_r \cup J_t)]} w_j,$$

or equivalently as

$$\sum_{j \in [J - (J_r \cup J_t)]} w_j < \sum_{j \in J_r} w_j - \sum_{j \in J_t} w_j$$

and

$$\sum_{j \in [J - (J_r \cup J_t)]} w_j < \sum_{j \in J_t} w_j - \sum_{j \in J_r} w_j.$$

In the last form, the inequalities are contradictory since the left sides of both are always non-negative and one of the right sides must be zero or negative. This completes the proof.

The single-facility rectangular distances location problem is given by

$$\text{minimize} \sum_{j=1}^{n} w_j \left[ |x_1 - a_{1j}| + |x_2 - a_{2j}| \right],$$

where the vectors $a_j = (a_{1j}, a_{2j})$, $j \in J$, are the locations of the fixed facilities, and $(x_1, x_2)$ is the facility location to be determined. Any rectangular distances location problem is decomposable into

two separate problems, one being the location along the $x_1$-axis, the other being the location along the $x_2$-axis. The $x_1$-axis location problem is given by

$$\text{minimize } g_1(x_1) = \sum_{j=1}^{n} w_j \, |x_1 - a_{1j}| .$$

$g_1(x_1)$ is convex and piece-wise linear. For this reason, the optimal solution must lie at one of the $a_{1j}$, $j \epsilon J$ or on either of two adjacent $a_{1j}$ and any point between. We first consider the case where the latter condition does not occur. Let $j^*$ be the value of $j \epsilon J$ such that $a_{1j*}$ is the optimal value of $x_1$. Then the following two inequalities must hold:

(2.1) $$\sum_{\substack{\text{all } j \\ \text{s.t. } a_{1j} \leqslant a_{1j*}}} w_j \; > \sum_{\substack{\text{all } j \\ \text{s.t. } a_{1j} > a_{1j*}}} w_j$$

and

(2.2) $$\sum_{\substack{\text{all } j \\ \text{s.t. } a_{1j} \geqslant a_{1j*}}} w_j > \sum_{\substack{\text{all } j \\ \text{s.t. } a_{1j} < a_{1j*}}} w_j .$$

Similarly, the $x_2$-axis location problem is given by

$$\text{minimize } g_2(x_2) = \sum_{j=1}^{n} w_j \, |x_2 - a_{2j}| .$$

If $a_{2j*}$ is the optimal value of the $x_2$, then the following two inequalities must hold:

(2.3) $$\sum_{\substack{\text{all } j \\ \text{s.t. } a_{2j} \leqslant a_{2j*}}} w_j \; > \sum_{\substack{\text{all } j \\ \text{s.t. } a_{2j} > a_{2j*}}} w_j$$

and

(2.4) $$\sum_{\substack{\text{all } j \\ \text{s.t. } a_{2j} \geqslant a_{2j*}}} w_j \; > \sum_{\substack{\text{all } j \\ \text{s.t. } a_{2j} < a_{2j}^*}} w_j .$$

For the case where $a_{1j*}$ is an alternative optimum then either (2.1) or (2.2) will be an equality. However, in this case the values of the $w_j$ can be perturbed by an arbitrarily small amount to produce a unique optimum at $a_{1j*}$ while leaving the optimal value of $g_1(x_1)$ effectively unaltered. Similar results hold for $a_{2j*}$. Thus it is sufficient to consider as candidates for optimality only points which can satisfy the necessary conditions for a point to be a unique optimum.

Since all the $w_j$ values are used once in each of the inequalities (2.1), (2.2), (2.3), and (2.4), we may write the four inequalities for any point that is a candidate for the optimal solution and apply Theorem 1. This is equivalent to comparing the left sides of Inequalities (2.1) and (2.2) with the left sides of Inequalities (2.3) and (2.4) pairwise and checking them for common elements. If no common elements are found in any single pair of inequalities, by Theorem 1, the point may be eliminated from the set of possible solution points. Define the sets $S_1$ and $S_2$ as

$$S_1 = \{a_{11}, a_{12}, \ldots, a_{1n}\}, \qquad S_2 = \{a_{21}, a_{22}, \ldots, a_{2n}\}.$$

The optimum solution of the single-facility rectangular distances location problem must belong to the set $S_1 \times S_2$. The algorithm given here consists of checking each point in the initial solution candidate set $S_1 \times S_2$ by the procedure given above, and eliminating any point which fails the test, thus producing a reduced solution candidate set $S$. A property of this test which makes the development of an efficient algorithm possible is that it does not depend on the magnitude of the $w_j$'s. For a multifacility location problem having no flows between the $p$ variable facilities, the problem is equivalent to $p$ single variable facility location problems with associated weights $w_{ij}$, $j = 1, \ldots, n$, for each problem. For each of these single location problems the set reduction algorithm will generate $S$ or a subset of $S$. Therefore $S$ contains all possible locations for the location-allocation problem (1.1) regardless of the magnitudes of the $w_{ij}$'s.

An example of a location problem with six fixed points is given in Figure 1. The initial solution candidate set consists of the intersections of all vertical and horizontal lines through the existing facilities. We will check all candidate points moving from left to right along the $x_1$-axis and for each of these lines, all intersections moving from bottom to top along the $x_2$-axis. For a point to be left in the solution candidate set it must satisfy all four of the left side inequality comparison tests. In this procedure both left sides of Inequalities (2.1) and (2.2) are compared with the left side of Inequality (2.3) and a O or I entered in the first line of Table 1. The left sides of Inequalities (2.1) and (2.2) are then compared with the left side of Inequality (2.4). For any value of $a_{2j}$ on the $x_2$-axis, both entries in the last column of Table 1 must be I's in order for the point to remain in the candidate solution set.

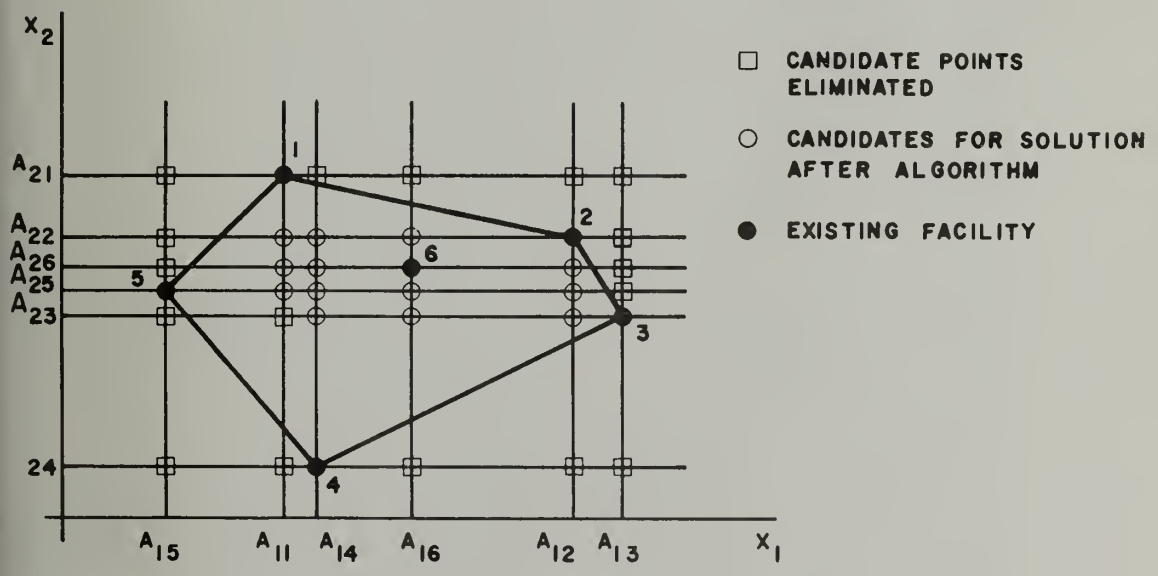


FIGURE 1

TABLE 1

| $x_1$ | Left sides of inequalities (2.1) and (2.2) | $x_2$ | Left sides of inequalities (2.3) and (2.4) | Pass test (I) Fail test (O) | Points in candidate set |
|---|---|---|---|---|---|
| $a_{15}$ | {5}<br>{1, 2, 3, 4, 5, 6} | $a_{24}$<br>" | {4}<br>{1, 2, 3, 4, 5, 6} | O<br>I | |
| | | $a_{23}$<br>" | {3, 4}<br>{1, 2, 3, 5, 6} | O<br>I | |
| | | $a_{25}$<br>" | {3, 4, 5}<br>{1, 2, 6, 5} | I<br>I | $(a_{15}, a_{25})$ |
| | | $a_{26}$<br>" | {3, 4, 5, 6}<br>{1, 2, 6} | I<br>O | |
| | | $a_{22}$<br>" | {2, 3, 4, 5, 6}<br>{1, 2} | I<br>O | |
| | | $a_{21}$<br>" | {1, 2, 3, 4, 5, 6}<br>{1} | I<br>O | |
| $a_{11}$ | {1, 5}<br>{1, 2, 3, 4, 6} | $a_{24}$<br>" | {4}<br>{1, 2, 3, 4, 5, 6} | O<br>I | |
| | | $a_{23}$<br>" | {3, 4}<br>{1, 2, 3, 5, 6} | O<br>I | |
| | | $a_{25}$<br>" | {3, 4, 5}<br>{1, 2, 6, 5} | I<br>I | $(a_{11}, a_{25})$ |
| | | $a_{26}$<br>" | {3, 4, 5, 6}<br>{1, 2, 6} | I<br>I | $(a_{11}, a_{26})$ |
| | | $a_{22}$<br>" | {2, 3, 4, 5, 6}<br>{1, 2} | I<br>I | $(a_{11}, a_{22})$ |
| | | $a_{21}$<br>" | {1, 2, 3, 4, 5, 6}<br>{1} | I<br>I | $(a_{11}, a_{21})$ |
| $a_{14}$ | {1, 5, 4}<br>{2, 3, 4, 6} | $a_{24}$<br>" | {4}<br>{1, 2, 3, 4, 5, 6} | I<br>I | $(a_{14}, a_{24})$ |
| | | $a_{23}$<br>" | {3, 4}<br>{1, 2, 3, 5, 6} | I<br>I | $(a_{14}, a_{23})$ |
| | | $a_{25}$<br>" | {3, 4, 5}<br>{1, 2, 6, 5} | I<br>I | $(a_{14}, a_{25})$ |
| | | $a_{26}$<br>" | {3, 4, 5, 6}<br>{1, 2, 6} | I<br>I | $(a_{14}, a_{26})$ |
| | | $a_{22}$<br>" | {2, 3, 4, 5, 6}<br>{1, 2} | I<br>I | $(a_{14}, a_{22})$ |
| | | $a_{21}$<br>" | {1, 2, 3, 4, 5, 6}<br>{1} | I<br>O | |

TABLE 1.—Continued

| $x_1$ | Left sides of inequalities (2.1) and (2.2) | $x_2$ | Left sides of inequalities (2.3) and (2.4) | Pass test (I) Fail test (O) | Points in candidate set |
|---|---|---|---|---|---|
| $a_{16}$ | {1, 4, 5, 6}<br>{6, 2, 3} | $a_{24}$<br>" | {4}<br>{1, 2, 3, 4, 5, 6} | O<br>I | |
| | | $a_{23}$<br>" | {3, 4}<br>{1, 2, 3, 5, 6} | I<br>I | $(a_{16}, a_{23})$ |
| | | $a_{25}$<br>" | {3, 4, 5}<br>{1, 2, 6, 5} | I<br>I | $(a_{16}, a_{25})$ |
| | | $a_{26}$<br>" | {3, 4, 5, 6}<br>{1, 2, 6} | I<br>I | $(a_{16}, a_{26})$ |
| | | $a_{22}$<br>" | {2, 3, 4, 5, 6}<br>{1, 2} | I<br>I | $(a_{16}, a_{22})$ |
| | | $a_{21}$<br>" | {1, 2, 3, 4, 5, 6}<br>{1} | O<br>I | |
| $a_{12}$ | {1, 2, 4, 5, 6}<br>{2, 3} | $a_{24}$<br>" | {4}<br>{1, 2, 3, 4, 5, 6} | O<br>I | |
| | | $a_{23}$<br>" | {3, 4}<br>{1, 2, 3, 5, 6} | I<br>I | $(a_{12}, a_{23})$ |
| | | $a_{25}$<br>" | {3, 4, 5}<br>{1, 2, 6, 5} | I<br>I | $(a_{12}, a_{25})$ |
| | | $a_{26}$<br>" | {3, 4, 5, 6}<br>{1, 2, 6} | I<br>I | $(a_{12}, a_{26})$ |
| | | $a_{22}$<br>" | {2, 3, 4, 5, 6}<br>{1, 2} | I<br>I | $(a_{12}, a_{22})$ |
| | | $a_{21}$<br>" | {1, 2, 3, 4, 5, 6}<br>{1} | O<br>I | |
| $a_{13}$ | {1, 2, 3, 4, 5, 6}<br>{3} | $a_{24}$<br>" | {4}<br>{1, 2, 3, 4, 5, 6} | I<br>O | |
| | | $a_{23}$<br>" | {3, 4}<br>{1, 2, 3, 5, 6} | I<br>I | $(a_{13}, a_{23})$ |
| | | $a_{25}$<br>" | {3, 4, 5}<br>{1, 2, 5, 6} | I<br>O | |
| | | $a_{26}$<br>" | {3, 4, 5, 6}<br>{1, 2, 6} | I<br>O | |
| | | $a_{22}$<br>" | {2, 3, 4, 5, 6}<br>{1, 2} | O<br>I | |
| | | $a_{21}$<br>" | {1, 2, 3, 4, 5, 6}<br>{1} | I<br>O | |

The algorithm has eliminated 17 of the original 36 solution candidate points.

## 3. MULTIFACILITY LOCATION

The candidate set reduction algorithm can accommodate the case for which weights are associated with distances between the variable facilities. It has been shown previously that the total cost function for locating $p$ facilities in relation to $n$ facilities is convex [19]. We define a "one at a time" algorithm as that which locates a single facility with all others held constant, then locates a second facility this way and proceeds similarly until no further improvement moves can be made. It has been shown by Wesolowsky and Love [26] that a "one at a time" algorithm will converge to optimum in a finite number of steps.

Assume a starting feasible solution is chosen with each new facility occupying one of the locations in $S$ and a "one at a time" optimizing procedure is used with all interfacility weights considered. At each iteration, when the new optimum location of any variable facility is determined (with the location of the $p-1$ remaining facilities held fixed), if the candidate set reducing procedure is applied, $S$ or a proper subset of $S$ will be arrived at as the set of possible optimal locations of the facility. Therefore, the optimum location of all new variable facilities must belong to $S$, even when weights between new facilities are considered.

## 4. A GEOMETRIC INTERPRETATION OF $S$: THE RECTANGULAR HULL PROPERTY

We now establish a geometrical interpretation of the set $S$ by using the concept of a rectangular hull. The rectangular hull $R$ of $\{a_1, \ldots, a_n\}$ has been defined by H. Juel [14]. It is the set of points $(x_1, x_2)$ which satisfy the following four conditions:

1. $x_1 \geq a_{1j}$ and $x_2 \geq a_{2j}$ for some $j \epsilon J$,
2. $x_1 \geq a_{1j}$ and $x_2 \leq a_{2j}$ for some $j \epsilon J$,
3. $x_1 \leq a_{1j}$ and $x_2 \leq a_{2j}$ for some $j \epsilon J$,
4. $x_1 \leq a_{1j}$ and $x_2 \geq a_{2j}$ for some $j \epsilon J$.

THEOREM 2: Any point $(x_1, x_2) \epsilon S_1 \times S_2$ that does not satisfy one of the four rectangular hull conditions will be rejected by the solution set reducing algorithm.

PROOF: Consider first a point which does not satisfy condition 1, but does satisfy conditions 2, 3, and 4 (if such a point exists). For some pair of values of $j$, say $j'$ and $j''$, where $j', j'' \epsilon J$, let this point be given by a $a_{j_1} = (a_{1j'}, a_{2j''})$, where $a_{j_1} \epsilon S_1 \times S_2$. Define the sets $S_3$ and $S_4$ as

$$S_3 = \{w_j, j \epsilon J \mid a_{1j} \leq a_{1j'}\},$$

$$S_4 = \{w_j, j \epsilon J \mid a_{2j} \leq a_{2j''}\}.$$

In order for $a_{j_1}$ to be an optimal solution to a single facility location problem then by the necessary conditions the following two conditions must hold:

$$\sum w_j \epsilon S_3 > \sum w_j \notin S_3$$

and

$$\sum w_j \epsilon S_4 > \sum w_j \notin S_4.$$

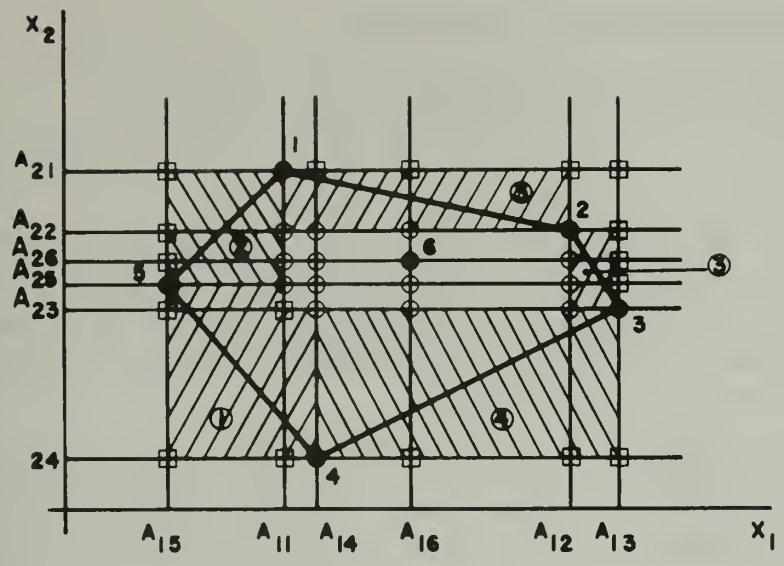


FIGURE 2

Since $a_{j_1}$ does not satisfy the first rectangular hull condition, for any $j \epsilon J$ such that $w_j \epsilon S_3$, it follows that $a_{2j} > a_{2j''}$. This means there are no $w_j$ common to $S_3$ and $S_4$, and $a_{j_1}$ would be eliminated by the set reduction algorithm. In a similar fashion it may be proven that any point in $S_1 \times S_2$ not satisfying any one of the rectangular hull conditions 2, 3, or 4 will be eliminated by the set reduction algorithm.

Points in $S_1 \times S_2$, which satisfy all but one of the rectangular hull conditions, have geometrical interpretations. The set of fixed points in Figure 1 is reproduced in Figure 2. A shaded area in Figure 2 denoted by ⓘ, $i = 1, 2, 3, 4$, indicates that any point in area $i$ satisfies all the rectangular hull conditions except for condition $i$. The shaded areas are each rectangles which are open on the two sides within the convex hull $C$ of $\{a_1, \ldots, a_n\}$. When these rectangles are considered in relation to the points in $S_1 \times S_2$ which remain as solution candidates after the set reduction algorithm, we may place the rectangles in four categories: lower left, denoted by ①; upper left, denoted by ②; upper right, denoted by ③; and lower right, denoted by ④. It is easily proven, of course, that any point not satisfying two or more of the rectangular hull conditions does not belong to $S$. It has been shown [25] that the initial solution candidate set $S_1 \times S_2$ can be reduced to $(S_1 \times S_2) \cap C$. However, in general

$$S = (S_1 \times S_2) \cap R \subset (S_1 \times S_2) \cap C$$

and the rectangular hull property produces a candidate set with a smaller number of points.

## 5. COMPUTATIONAL PROCEDURE

Let $X_p^*$ be the optimal set of $p$ points $\chi_1^*, \ldots, \chi_p^*$ for problem (1.1). Then it is easily shown that the optimal allocations are given by

$$w_{ij}^* = \begin{cases} w_j, & \text{if } d(\chi_i, a_j) = \min\limits_{\chi_k \epsilon X_p^*} \{d(\chi_k, a_j)\} \\ 0, & \text{otherwise.} \end{cases}$$

Problem (1.1) then reduces to finding $X_p^* \subset S$ such that

$$R(\chi^*, w^*) = \sum_{j=1}^{n} w_j d(X_p^*, a_j) \leq \sum_{j=1}^{n} w_j d(X_p, a_j),$$

where $X_p \subset S$ and $d(X_p, a_j) = \min \{d(\chi_1, a_j), \ldots, d(\chi_p, a_j)\}$.
This is equivalent to Hakimi's statement [11] of the problem of finding the $p$-median of a weighted connected graph $G$, where $S$ is the set of all vertices of $G$. The $w_j$'s represent weights of vertices in $S$ which are existing facility locations. All other vertices have zero weights. This equivalence allows for the solution of problem (1.1) by any technique for solving the $p$-median problem once the candidate set reduction algorithm has been used to define $S$.

Such techniques typically use some form of implicit enumeration [6, 12, 13] where computational efficiency may be enhanced by providing a "sharp" initial upper bound on the solution of the problem. To this end it is suggested that a "reduced" problem be initially solved by using $S = \{a_1, \ldots, a_n\}$ to yield $R(\hat{\chi}, \hat{w})$, say, which provides an initial upper bound for the solution to the full problem. The rationale is that $X_p^*$ often includes one or more existing facility locations and relative to the full problem the reduced problem takes very little time to solve.

## 6. COMPUTATIONAL EXAMPLES

Computational results have been generated on a Univac 1110 computer using a relatively efficient backtrack programming approach to solving the $p$-median problem given by Hung [12]. To indicate effects of existing facility locational configurations on the problem solution, the eight example problems given by Cooper [2] with $n = 7$, $p = 2$ and $w_j = 1$ for all $j$ were solved using rectangular distances. The results are presented in Table 2, where NCP is the number of points in $S$ and NEF is the number of existing facility locations $a_j$ in $X_p^*$. Alternative optima were frequently encountered.

TABLE 2

| Problem | $R(\chi^*, w^*)$ | NCP | Elements of $X_2^*$ | NEF |
|---|---|---|---|---|
| 1 | 59 | 26 | (16, 14), (22, 27) | 0 |
| 2 | 72 | 9 | (20, 8), (20, 32) | 2 |
| 3 | 41 | 14 | (5, 26), (35, 26) | 2 |
| 4 | 55 | 13 | (15, 23), (21, 32) | 2 |
| 5 | 48 | 21 | (11, 20), (24, 17) | 2 |
| 6 | 51 | 7 | (13, 24), (27, 17) | 2 |
| 7 | 75 | 21 | (18, 19), (25, 33) | 0 |
| 8 | 74 | 16 | (28, 6), (33, 23) | 2 |

Results for randomly generated problems with $p = 2$ and 3 for several values of $n$ are presented in Table 3. The CPU time in minutes for generating $S$ and for solving the resultant $p$-median problem is TCP and TPM, respectively.

TABLE 3

| $n$ | $p$ | NCP | NEF | TCP (min.) | TPM (min.) | $n$ | $p$ | NCP | NEF | TCP (min.) | TPM (min.) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 2 | 78 | 2 | 0.004 | 0.052 | 12 | 3 | 78 | 3 | 0.004 | 0.496 |
| 14 | 2 | 82 | 0 | 0.005 | 0.061 | 13 | 3 | 89 | 1 | 0.005 | 0.574 |
| 16 | 2 | 140 | 0 | 0.008 | 0.282 | 14 | 3 | 82 | 2 | 0.005 | 0.588 |
| 18 | 2 | 214 | 1 | 0.012 | 0.986 | 15 | 3 | 133 | 2 | 0.007 | 4.424 |
| 20 | 2 | 234 | 0 | 0.013 | 1.271 | 16 | 3 | 140 | 1 | 0.008 | 6.026 |
| 25 | 2 | 462 | 0 | 0.026 | 9.589 | 17 | 3 | 207 | 1 | 0.011 | 13.942 |
| 30 | 2 | 703 | 0 | 0.040 | 33.213 | 18 | 3 | 214 | 2 | 0.012 | 24.170 |
| 35 | 2 | 984 | 1 | 0.057 | 91.332 | 19 | 3 | 234 | 1 | 0.013 | 41.863 |

The problem for $n=12$ was solved for $p=4, 5, \ldots, 10$ with TPM $=0.85, 1.66, 2.16, 0.84, 0.66, 0.68$, and 0.72, respectively. A total of 20 $n=12$, $p=3$ randomly generated problems were solved yielding an average NCP $=77.85$ with a low of 55 and a high of 97. Two problems had NEF $=0$, 10 had NEF $=1$, six had NEF $=2$, and two had NEF $=3$. To further investigate the possible merit of initially solving the reduced problem, values of $[R(\hat{\chi}, \hat{w}) - R(\chi^*, w^*)]/R(\chi^*, w^*)$ were computed. The average was 0.034 with a high of 0.118.

It can be seen that the efficiency of the procedure presented in this paper for solving rectangular distance location-allocation problems depends solely on the efficiency of the method used to solve the equivalent p-median problem. As the efficiency of such methods increases, the computational times presented here will decrease. The randomly generated data and solutions for two of the test problems are as follows.

1. $(a_{1j}, a_{2j}, w_j)$, $j=1, \ldots, 16$, for $n=16$, $p=3$:

(9964, 8900, 9926), (1473, 1244, 8963), (6636, 1290, 5226), (8772, 531, 130), (8472, 485, 5370), (9355, 5943, 8661), (1312, 723, 6997), (6262, 919, 2396), (1396, 646, 5149), (4933, 5299, 9575), (9364, 7293, 7265), (7208, 9323, 2838), (4256, 104, 7185), (5326, 7630, 3225), (7512, 4919, 9149), (5929, 3262, 4177).

$$X_3^* = \{(9364, 7293), (1473, 723), (6636, 4919)\} \text{ and } R(\chi^*, w^*) = 191354085.$$

2. $(a_{1j}, a_{2j}, w_j)$, $j=1, \ldots, 30$, for $n=30$, $p=2$:

(9257, 4297, 4322), (4978, 6367, 7633), (7334, 7629, 253), (2301, 9803, 7111), (71, 1009, 3509), (4994, 5102, 7871), (5719, 3217, 8678), (1532, 3507, 1608), (899, 5082, 2877), (8202, 2152, 4781), (4681, 4362, 9431), (9575, 9337, 6699), (7461, 3420, 2353), (7793, 6611, 2400), (2643, 7710, 9623), (8644, 5652, 8326), (9122, 3052, 3270), (2445, 7944, 8813), (2557, 9716, 4252), (7721, 5259, 1360), (3457, 1298, 132), (5684, 8939, 8717), (59, 6403, 3564), (3516, 5972, 5760), (4008, 1123, 9568), (4446, 9875, 9173), (6837, 320, 9614), (2629, 4372, 7930), (5246, 5702, 498), (831, 7878, 2572).

$$X_2^* = \{(2643, 8939), (4994, 4362)\} \text{ and } R(\chi^*, w^*) = 516254969.$$

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# APPROXIMATIONS FOR THE CONSOLIDATION OF MULTIECHELON REPAIR FACILITIES*

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**ABSTRACT**

This paper develops and illustrates an approximate approach for analytically assessing the impacts on both costs and service of consolidation of repair facilities. The repair facilities are two echelon generalizations of the classical repairmen problem in which two types of failures, say major and minor, can occur, each type requiring repair at a different echelon. The questions addressed are the reductions possible in spares, repairmen, and service rates due to the consolidated system's increased efficiency, as well as the physical separation between the users and the consolidated repair facility that is economical. The method of analysis is based upon asymptotic approximations developed for the repairmen problem, valid when the number of operational equipments is large (greater than 50); it is helpful since it provides a tractable means for predicting the steady-state performance of the decentralized and consolidated installations as a function of the many parameters involved without having to resort to an exhaustive computation of all the exact steady-state probabilities.

## 1. INTRODUCTION

This paper develops a tractable methodology for investigating the impacts on both costs and service levels of consolidation or centralization of repair activities. In particular, we shall consider consolidation of a system in which all the resources (spares, repairmen, repair parts, etc.) previously owned and managed by several disjointed facilities, with no mechanisms for sharing, are combined in one facility. Such a consolidation obviously removes the imbalance of idle repair capability existing at one facility and at the same time excessive delays occurring at other facilities. However, the price clearly paid for this consolidation will be longer turn-around time due to the decreased proximity of the repair facilities to the users.

Among the key questions to be answered are:

(1) Given the consolidated facility has the same resources as available under the disjointed configuration, what is the increased level of service (in terms, for example, of meeting some minimum operational requirements) that can be provided.

(2) What reduction in spares, repairmen, or repair rates can be tolerated under the consolidated arrangement and still provide the same level of service.

(3) What kind of separation between the users and the consolidated facility can be bought due to the increased efficiency of the consolidated system?

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## 2. DISCUSSION OF MODEL

The repair facilities considered in this paper are two-echelon generalizations of the so-called classical repairmen problem (for example, see Barlow and Proschan [1]) for which it is assumed:

(1) There is an operational requirement for $N$ equipments to be functioning continuously (otherwise the system's performance is degraded). In addition, the equipments are all assumed to be stochastically independent of one another and fail according to a Poisson distribution with a mean time between failure of $1/\lambda$.

(2) Backing up these $N$ operational equipments are $M$ spares, which can be used to fill any of the operating slots on an as-needed basis.

(3) Two types of failure are considered, called major and minor (in general, the approach can be easily extended to cover an arbitrary number of failure types). With probability $p$, a failure results in a requirement for service from Repair Echelon 1 (perhaps a local repair facility or tender); with probability $1 - p$ the failure is more serious and can be repaired* only at Repair Echelon 2.

(4) Both repair facilities are capable of repairing only a fixed number of units simultaneously, $S_1$ for Echelon 1 and $S_2$ for Echelon 2. If all repairmen are busy at a given Echelon, the failed item joins the waiting line at the Echelon and waits until a repairman is free. In addition, the repair times at each Echelon are also independent, identically distributed exponential random variables with means $1/\mu_1$ for Echelon 1 and $1/\mu_2$ for Echelon 2.

(5) Finally, when repairs are completed, they return to the spare pool and are once again available to fill operational openings. The flow is as depicted on the following page (Figure 1).

The method of analysis utilized to analyze the impacts of consolidation relies directly upon the application of approximations developed earlier (see Iglehart and Lemoine [3]). Several approximations were obtained by noting that when both the failure and service times have exponential distributions, the stochastic processes representing the number of units waiting or undergoing repair at each of the

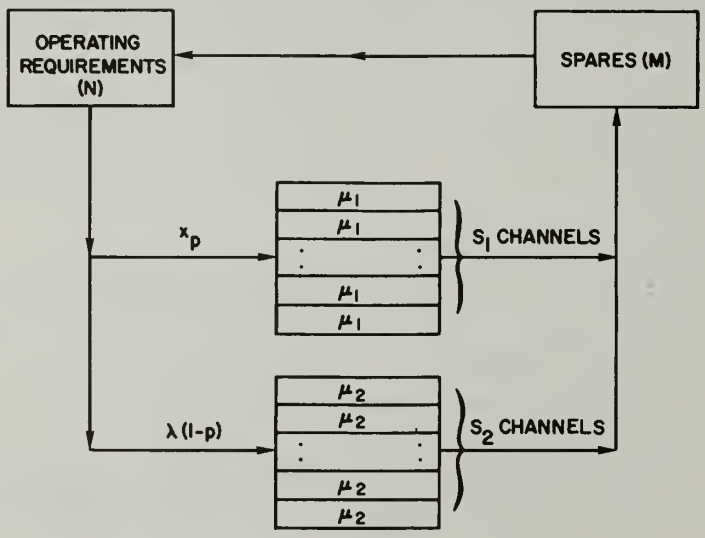


FIGURE 1

* An alternate interpretation of the repair activity at Repair Echelon 2 could be one-for-one ordering. This would be appropriate if the major failure were catastrophic.

two repair facilities is a finite state, continuous parameter Markov Chain, which attains a stationary distribution.

To be more specific, let $\mathbf{X}_N(t)$ denote the vector $(X_N^1(t), X_N^2(t)$ where $X_N^i(t)$ represents the number of units waiting and undergoing repair at facility $i$ at time $t$, and $N$ is the number of operational slots. Then $\mathbf{X}_N(t)$ is a positive recurrent Marcov chain with a finite state space $\Delta_N = \{(i, j): i, j \geq 0, i+j \leq N+M\}$, where $M$ is the number of spares. Then as is shown by Iglehart and Lemoine [2], $\{\mathbf{X}_N(t): t \geq 0\}$ is a reversible competition-process which allows them to define the stationary distribution $\mathbf{X}_N$, and further to prove a variety of propositions regarding the stationary distribution of $\mathbf{X}_N$, as $N$, the number of operational slots, becomes large. These theorems are broken into various cases of light and heavy traffic and depend upon the values of the eight parameters involved, namely, $N$, $M$, $\lambda$, $\mu_1$, $\mu_2$, $p$, $S_1$ and $S_2$. The key mathematical results of interest for the applications described in this note are the following:

let $a_1 = \dfrac{\lambda p}{M_1}$ and $a_2 = \dfrac{\lambda(1p)}{M_2}$

$$S_1 \sim N \cdot s_1, \qquad 0 < s_1 < 1 \qquad \text{as } N \to \infty$$

$$S_2 \sim N \cdot s_2, \qquad 0 < s_1 < 1 \qquad \text{as } N \to \infty$$

$$M \sim N \cdot m, \qquad m > 0 \qquad \text{as } N \to \infty$$

then if any one of the conditions (a), (b), (c), or (d) hold where condition (a) is

$$s_1 + s_2 < m, \qquad a_1 < s_1, \qquad a_2 < s_2;$$

condition (b) is

$$a_1 < s_1 \leq m, \qquad a_2 < ms_2;$$

condition (c) is

$$a_1 < m < s_1, \qquad a_2 < m < s_2;$$

and condition (d) is

$$s_1 + s_2 > m, \; a_1 < s_1 \leq m, \qquad a_2 < s_2 \leq m$$

and either

$$m > s_1 + a_2 \qquad \text{or} \qquad m > s_2 + a_1;$$

then,

$$(\mathbf{X}_N - N\mathbf{a})/N^{1/2} \Rightarrow N(\underset{\sim}{\mathbf{O}}, \underset{\sim}{\Sigma}), \qquad \text{where} \qquad \underset{\sim}{\Sigma} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix};$$

i.e., if there are a sufficient number of spares and both facilities are in light traffic, then for large $N$, $X_N^{(1)}$ and $X_N^{(2)}$ are well approximated by independent, normally distributed random variables with means $Na_1$ and $Na_2$, respectively, and standard deviations of $N^{1/2}a_1$ and $N^{1/2}a_2$, respectively.

## 3. NUMERICAL ILLUSTRATION

A numerical example is included illustrating the approach for one combination of the parameters. While the example, for ease of presentation, addresses the impact of consolidating two identical, but

separate facilities, there is no conceptual difficulty in dealing with an arbitrary number of nonidentical facilities. Also, it might be well to point out that calibration studies have shown the asymptotic results to be quite accurate, i.e., within 2 percent, when the number of operating units is larger than 50.

Consider the following set of parameters, describing the characteristics of each of two identical disjointed repair installations, each installation consisting of a two-echelon repair facility. Let:

(1) $N$, The number of operational requirements for each, equal 75;

(2) $M$, the number spares for each installation, equal 13;

(3) $1/\lambda$, the mean time between equipment failures, equals 66.7 days;

(4) $p$, the probability of an equipment failure being minor, equals 9 (i.e., capable of being repaired at the lower echelon);

(5) $1-p$, the probability of a failure being major, equals 0.1 (i.e., requiring repair or replacement at the higher echelon);

(6) $S_1$, the number of lower echelon service channels in each of the two disjointed installations be 6;

(7) $s_2$, the number of higher echelon service channels in each of the two disjointed installations be 3;

(8) $1/\mu_1$, mean repair time (not including waiting time) at each lower echelon, equal 5 days;

(9) $1/\mu_2$, mean repair time at each higher echelon, be 20 days.

(10) $X_N^{(1)}(1/\lambda, p, 1/\mu_1)$ Denote the steady-state number of units at a single installation awaiting or undergoing repair at the lower echelon, i.e., echelon 1, as a function of $\lambda$, $p$, $M$, and $N$.

(11) $X_N^{(2)}(1/\lambda, p, 1/\mu_2)$ Denote the steady state number of units at a single installation awaiting or undergoing repair at the higher echelon, i.e., echelon 2, as a function of $\lambda$, $p$, $M_2$ and $N$.

Then, by using the notation developed in section 2, we first note that by considering each installation separately, $s_1$ equals 6/75, $s_2$ equals 3/75, and $m=13/75$, and that indeed condition (a) holds. Also note that if the two disjointed installations, with no sharing of spares or repairmen, are replaced by one consolidated installation, having available to it the combined resources, i.e., 176 equipments, 12 minor service channels, and 6 major service channels, then condition (a) again automatically holds.

Then by using the results of section 2, one finds:

(a) Under the scenario in which the two identical repair installations are kept disjoint (with no sharing of resources allowed), the likelihood that they can both simultaneously meet their operational requirement of 75 is given by

$$\{P[X_{75}^{(1)}(66.7, 0.9, 5) + X_{75}^{(2)}(66.7, 0.9, 20) \leq 13]\}^2 \quad \text{or} \quad 0.83.$$

In contrast, under the consolidated scheme, we find that the probability of maintaining an equivalent number of operational units, i.e., at the centralized facility is given by

$$P[X_{150}^{(1)}(66.7, 0.9, 5) + X_{150}^{(2)}(66.7, 0.9, 20) \leq 26)] \quad \text{or} \quad 0.998,$$

thereby providing about a 22-percent gain in the level of protection with the same repair resources.

(b) If one holds the level of protection constant, one finds that consolidation can bring about a reduction of four spares for a 17-percent reduction, and still yield the same level of protection. Mathematically, this means that

$$P\{X_{150}^{(1)}(66.7, 0.9, 5) + X_{150}^{(2)}(66.7, 0.9, 20) \leq 22\} \geq 0.83.$$

Also, in the case where only 95 percent of the operational sl ots need to be filled, then a similar computation shows that consolidation can yield a 25-percent reduction in spares.

(c) Under the consolidated arrangement with the same total number of spares available a 19-percent degradation in the service rate at the first echelon (i.e., $1/\mu_1$ can increase from 5 days to 5.9 days), and a 33-percent degradation of the service rate at the second echelon (i.e., $1/\mu_2$ can increase from 20 days to 26.6 days) can be tolerated and still yield at least the same level of protection as in the decentralized scheme. Mathematically, this is expressed as:

$$P\{X_{150}^{(1)}(66.7, 0.9, 5.9) + X_{150}^{(2)}(66.7, 0.9, 26.6) \leq 26\} \geq 0.83.$$

(d) As has been mentioned earlier, a key disadvantage of centralization is that it is not possible for the central facility to be close to all users; hence, for the same level of service to be provided more pipeline inventory is required. For the case specified, it can be shown that the increased efficiency afforded by the consolidation permits an additional separation (between the users and the consolidated activity) of about 0.9 days shipping time. In other words, the increased average pipeline requirements due to the roundtrip additional shipping time of 1.8 days from the users to the consolidated facility back to the user is offset by the reduction in spares afforded by the increased efficiency of the consolidated arrangement. This follows as a simple application of Palm's Theorem which states that if the demand is Poisson with parameter $\alpha$ and the average pipeline time is $\bar{L}$, then the average pipeline inventory is simply $\alpha\bar{L}$. For the case considered, Iglehart and Lemoine show if $Y_N = N - [X_N^{(1)} + X_N^{(2)} - M]^+$, i.e., the number of operational slots filled, then $\lim_{N\to\infty} P\{Y_N = N\} = 1$. Hence, the rate at which equipments are breaking down in each facility is approximately $N$. Hence, if the consolidated facility were to be located midway between the two groups of users, such that the additional one-way shipping time from the user to the repair facility was $\Delta$ units of time, then the additional total average pipeline inventory required to take into account this increased shipping time would be approximately $4N\lambda\Delta$. This follows since for both sets of users, the failed items must be shipped from the user's area to the repair facility and then the repaired item is returned to the user. Hence, by setting $4N\lambda\Delta$ equal to the reduction in spares achievable, i.e., four for the case considered, we have the value of $\Delta$.

(e) In a similar manner one can compute such statistics as the increase in the failure rate $\lambda$, the change in the likelihood $p$ that a failed item will require major repair, or within limits the change in $N$ the number of equipments that can be tolerated under the consolidated scheme and still yield the same protection as afforded by the decentralized scheme.

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# A DYNAMIC, NONSTATIONARY INVENTORY PROBLEM FOR A PRICE/QUANTITY SETTING FIRM

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**ABSTRACT**

A dynamic and nonstationary model is formulated for a firm which attempts to minimize total expected costs over a finite planning horizon. The control variables are price and production. The price $p$ and the demand $\xi$ are linked through the relationship $\xi = g(p) + \eta$, where $g(p)$ is the riskless demand curve and $\eta$ is a random variable. The general model allows for proportional ordering costs, convex holding and stockout costs, downward sloping riskless demand curve, backlogging, partial backlogging, lost sales, partial spoilage of inventory, and two modes of collecting revenue. Sufficient conditions are developed for this problem to have an optimal policy which resembles the single critical number policy known from stochastic inventory theory. It is also shown what set of parameters will satisfy these sufficiency conditions.

## 1. INTRODUCTION

We will present a periodic, dynamic, nonstationary inventory model where the firm makes a pricing and a production decision in each period. The demand for the product can be written $\xi = g(p) + \eta$ where $\xi$ is the demand, $g(p)$ is the expected demand curve, $p$ is the price charged, and $\eta$ is a random variable with known probability distribution $\varphi(\cdot)$ and cummulative distribution function $\Phi(\cdot)$.

In order to place this problem in a wider setting, it is helpful to use the concept of behavioral mode introduced by Leland [9]. According to him there are, in general, at least three types of firms which warrant consideration, namely the quantity setting firm, the price setting firm, and the price/quantity setting firm. Demand and price are linked through some specified relationship $f(\xi, p, \eta) = 0$. While the price/quantity setting firm makes both a price and a production decision before the value of $\eta$ is realized, the price setting firm can postpone the production decision and the quantity setting firm can postpone the price decision until the value of $\eta$ is known.

Most works in inventory theory on multiperiod problems have been concerned with the special case where the expected demand curve is infinitely elastic [1, 3, 6, 7, 12, 13, 14, and 15]. In these models the price is not a variable and the firm they model can most conveniently be characterized as a quantity setting firm. The abundance of such models contrasts sharply with the scarcity of inventory models for the price setting firm, the quantity setting firm with a downward sloping expected demand curve and the price/quantity setting firm.

Our contribution is to the theory of the price/quantity setting firm. Other results in this area can be found in [4, 5, 8, 10, 11, 16], and [17]. In most studies, the random variable $\eta$ is introduced into the

demand process in a multiplicative fashion or in an additive fashion as in our case.* Although more complex relationships linking the demand, the price and the random variable are certainly possible, they are difficult to analyze.

Either one-period problems or infinite stage steady-state problems are studied by Hempenius [5], Karlin and Carr [8], Mills [11], and Zabel [16]. Miercort [10] presents an $n$-period inventory model where backlogging is allowed. The demand has a multiplicative character. However, it is not clear what parameter values will satisfy his sufficiency conditions for the optimal policy to have the required form. Furthermore, expected profit is defined such that the backorders are met at a zero price, which greatly simplifies his analysis at the expense of realism. Ernst [4] and Zabel [17] both analyze multiperiod inventory models with additive demand. They both assume that excess demand is lost (lost sales case). Zabel assumes a convex ordering cost function, a concave demand curve and linear holding and stockout costs. Ernst assumes that the demand curve as well as all costs are linear. On the other hand, Ernst allows for a wider range of distributions with his $PF_2$ assumptions, while Zabel confines his treatment to the exponential and the uniform distribution.† Zabel also investigates the case where the demand is multiplicative.

The firm which is being modeled here has the following operating characteristics. At the beginning of each period the firm will decide what price to charge and how much to produce in the period. Production is assumed to be instantaneous and nonnegative. After the inventory has been increased to the desired level, the demand occurs. The firm meets this demand as long as there are units in the inventory. If there is still some unsatisfied demand, there are two possibilities. Demand can be backordered or it can be lost. We will treat both of these extreme cases as well as the case of partial backlogging. The sales price for the backordered demand will be the price which was in effect at the time the demand was backlogged.

Regarding the collection of revenue, there are at least two options which should be considered. One possibility would be to assume that the customers pay as soon as they place their orders with the firm or possibly a fixed length of $\lambda$ periods later. For instance, most mail order businesses operate with $\lambda = 0$, the payment accompanies the order. Another possibility would be to assume that the customers pay $\lambda$ periods after the time the merchandise is delivered. Both cases will be discussed. This distinction is important only when backlogging is permitted, since the two cases coincide when all excess demand is lost.

We will assume proportional ordering costs and convex holding and stockout costs. For each period of the planning horizona choice can be made as to the exact form of the cost functions, the form of the expected demand curve and the probability distribution, the fraction of inventory which will deteriorate in the period, the fraction (if any) of excess demand which will be backlogged and the way the revenue is collected. Since all these decisions only affect the parameters of the model, it is possible to analyze all the possible models jointly in a common framework which we have called the general $n$-period problem.

In order to simplify the notation, the results will be proved only for the stationary case. However, all the results can be generalized without difficulty to the nonstationary case.

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*The demand is said to be multiplicative if $\xi = \eta \cdot g(p)$ where $\eta$ is a nonnegative random variable having expected value $E(\eta) = 1$.

†$PF_2$ includes many of the common distributions used in applications such as the exponential, the uniform, the normal and the truncated normal.

## 2. SUMMARY OF RESULTS

The optimal policy for the general $n$-period problem resembles the critical number policy of classical inventory theory. Let period $n$ be the current period and let period one be the last period in the planning horizon. Then it is demonstrated that the optimal policy is defined by a sequence of order up to points $y_1^*, y_2^*, \ldots, y_n^*$, a sequence of prices $p_1^*, p_2^*, \ldots, p_n^*$ and another sequence of prices $p_1^*(x_1), p_2^*(x_2), \ldots, p_n^*(x_n)$ which is dependent on the inventory levels $x_1, \ldots, x_n$. If the inventory level $x_n$ at the beginning of period $n$ is below $y_n^*$, an order for the difference is placed and the price $p_n^*$ is charged. If $x_n$ exceeds $y_n^*$, then it is optimal not to order in the period and to charge the price $p_n^*(x_n)$.

In order to prove the optimality of this policy, we have to make certain assumptions. Theorem 1 limits the range of possible parameter values, such that the general $n$-period problem becomes well posed. The other assumptions in Theorems 2 and 3 do not have any straightforward economic interpretation and they will in some cases be difficult to verify.

However, the theorems will be satisfied if we assume a linear expected demand curve, linear stockout costs, convex holding costs, and that $\varphi(\cdot)$ is a $PF_2$ distribution. In the particular case where all excess demand is backlogged, the demand curve is concave and the revenue is collected $\lambda$ periods after the time the orders are placed with the firm, Theorems 2 and 3 will always be satisfied. No additional assumptions have to be made concerning the form of the cost functions and the probability distribution. Furthermore, in this case the optimal price $p_n^*$ is independent of the moments of $\varphi(\cdot)$ and equal to that price which equates marginal discounted revenue with respect to quantity with marginal ordering cost in a deterministic one period problem obtained by setting $\eta \equiv 0$. Without loss of optimality, the pricing decision can be made separately from the quantity decision.

Ernst's results fall out as a special case of the general $n$-period problem. For his specific model where excess demand is lost, the solution of the general $n$-period problem shows that his results can be generalized to the nonstationary case and the case of convex holding costs. Ernst does not impose a restriction on the demand process such that the demand will always be nonnegative. In order to give his model economic meaning, he makes the assumption that if the demand is negative the firm will absorb this amount at the price it has selected. His model produces the result that the optimal price will not always be a decreasing function of starting inventory. We show that when the necessary assumptions are made to avoid the possibility of negative demand, the optimal price will always be a decreasing function of starting inventory.

## 3. THE GENERAL $n$-PERIOD PROBLEM

### Model Formulation

We will consider a class of discrete review price/production models in which the demand for a single product is given by $\xi = g(p) + \eta$. In order to avoid the possibility of negative demand, let $p_0$ be the highest price the firm may charge. Figure 1 shows the demand as a function of price where $\underline{\eta}$ and $\bar{\eta}$ are the lower and upper bounds on the random variable.

We have nothing to say about problems where $\underline{\eta} \to -\infty$ since, in this case, we cannot rule out the possibility of negative demand. In general, the more negative $\underline{\eta}$ is, the lower $p_0$ will be for a given expected demand curve. Although the use of distributions like the normal always admits the possibility of negative demand and therefore gives an inaccurate description of the demand process, such dis-

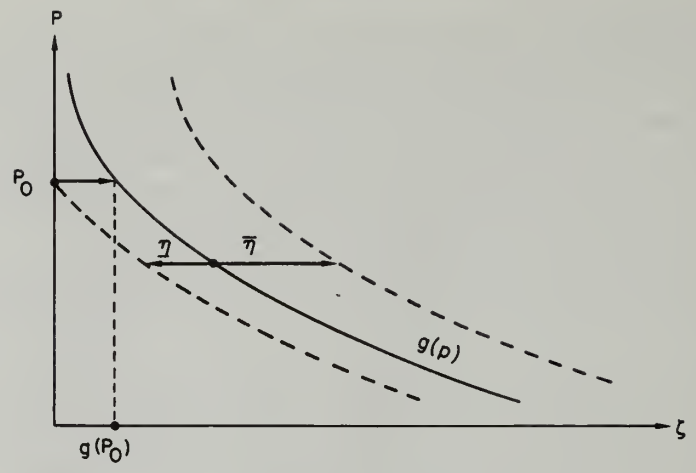


FIGURE 1. Demand as a function of price.

tributions may yet be helpful as an approximation device because of their computational advantages. We will assume that the expected demand curve $g(p)$ is downward sloping, that the probability density function $\varphi(\cdot)$ is continuous and strictly positive except possibly at the end points $\underline{\eta}$ and $\bar{\eta}$, and that the expected value of the random variable $\eta$ is equal to zero.

Let $x$ be the starting inventory and define $y$ to be the order up to point where $y$ is constrained to be larger than or equal to $x$. Assume that the holding cost function $h(y-\xi)$ and the stockout cost function $v(\xi-y)$ are continuous, convex and twice differentiable throughout the respective domains except on a set of measure zero. If $y \leq \xi$, then $h(y-\xi) \equiv 0$ and if $y \geq \xi$, then $v(\xi-y) \equiv 0$. Let $e$ and $f$ be parameters which can produce model variations, $t$ the fraction of unsatisfied demand which is backlogged, $u$ the fraction of inventory which spoils from one period to the next, and $\alpha$ a discount factor which is the value at the beginning of this period of one cost unit (e.g., a dollar) at the beginning of the next period. We will also assume that $0 \leq t, u, \alpha \leq 1$.

In order to preserve the similarity with the approach most often used in classical inventory theory. the problem will be formulated as a cost minimization problem where revenues are expressed as negative costs. Define $F_n(x)$ to be the expected costs incurred in the periods $n, n-1, \ldots, 1$ given that the starting inventory in period $n$ is $x$ and an optimal policy is followed thereafter. It turns out that the subsequent analysis is greatly facilitated by letting $y$ be substituted by a new variable $z$ defined as $z \equiv y-g(p)$. Let $L_n(z)$ consist of the expected holding and stockout costs in period $n$ as well as the net cost of the remaining periods in the planning horizon when an optimal policy is followed for these periods. In particular,

$$(3.1)\quad L_n(z) = \int_{\underline{\eta}}^{z} h(z-\eta)\varphi(\eta)d\eta + \int_{z}^{\bar{\eta}} v(\eta-z)\varphi(\eta)d\eta + \alpha \int_{\underline{\eta}}^{z} F_{n-1}[(1-u)(z-\eta)]\varphi(\eta)d\eta + \alpha \int_{z}^{\bar{\eta}} F_{n-1}[t(z-\eta)]\varphi(\eta)d\eta.$$

If $n=1$, then $F_0(\cdot)$ is the cost of filling the remaining backorders or salvaging the inventory at the end of the planning horizon. Let $c$ be the unit ordering cost and $s$ the unit salvage value, such that

$s \leq c$. Then, $F_0[(1-u)(z-\eta)] = -s(1-u)(z-\eta)$ and $F_0[t(z-\eta)] = -ct(z-\eta)$.

If $t=0$, we have the lost sales case where excess demand is lost; if $t=1$, backlogging is allowed. If $u=0$, we have the case of nonperishable goods while $u=1$ leads to the case of totally perishable goods. The model allows for partial backlogging and partial deterioration of inventory when $0 < t < 1$ and $0 < u < 1$.

In the case a positive amount $y-x$ is ordered in period $n$, the cost of ordering is equal to $C = cy - cx = cz + cg(p) - cx$.

The expected revenue in period $n$ will consist of the discounted price times the expected demand minus the discounted expected value of foregone revenue. Let $f$ be the multiperiod discount factor. In the case of $\lambda$ periods, $f = \alpha^{\lambda}$. We will also introduce a parameter $e$ which can be interpreted as the discounted fraction of excess demand which contributes to the amount of foregone revenue.

If the revenue is collected $\lambda$ periods after the time the orders are placed with the firm, then $e = (1-t)f$. If the backlogged demand in a period is met the next period and the revenue is collected $\lambda$ periods after the time the orders are satisfied, then $e = (1-\alpha t)f$. Let $R$ be the expected revenue in period $n$.

$$R = \begin{cases} fpg(p) - ep\displaystyle\int_z^{\bar{\eta}} (\eta - z)\varphi(\eta)\,d\eta & \text{if } z + g(p) \geq 0 \\ (f-e)pg(p) & \text{if } z + g(p) < 0. \end{cases}$$

Define $G_n(z, p) - cx$ to be the total expected costs given that the policy $(x, p)$ is chosen in period $n$ and an optimal policy is followed thereafter.

$$G_n(z, p) = C + cx + L_n(z) - R$$

$$= \begin{cases} cg(p) + cz + L_n(z) + ep\displaystyle\int_z^{\bar{\eta}} (\eta - z)\varphi(\eta)\,d\eta - fpg(p) & \text{if } z + g(p) \geq 0 \\ cg(p) + cz + L_n(z) - (f-e)pg(p) & \text{if } z + g(p) < 0. \end{cases} \tag{3.2}$$

The largest possible price the firm may charge is $p_0$. The general $n$-period problem can then be expressed as

$$F_n(x) = \min_{\substack{z \geq x - g(p) \\ p \leq p_0}} G_n(z, p) - cx. \tag{3.3}$$

We will call (3.3) the price-production constrained (PP-constrained) problem. Initially we will study the simpler problem

$$\min_{p < p_0} G_n(z, p), \tag{3.4}$$

which we refer to as the price constrained (P-constrained) problem.

**Optimal Output and Pricing Policy**

Under the assumption that $F_{n-1}(z)$ is differentiable and $\varphi(\eta)$ is continuous for $\eta \epsilon (\underline{\eta}, \bar{\eta})$, $G_n(z, p)$ is differentiable. $\phi(\eta)$ may be discontinuous at $\eta = \underline{\eta}$ and $\bar{\eta}$ without destroying the differentiability of $G_n(z, p)$. We conclude that $G_n(z, p)$ is once differentiable for $z$, $p \epsilon (-\infty, \infty)$.

The first order conditions for an optimal solution to the P-constrained problem are:

$$\partial/\partial z G_n(z, p) = 0 = \begin{cases} c + L_n'(z) & \text{if } z + g(p) < 0 \\ c + L_n'(z) - ep(1 - \Phi(z)) & \text{if } z + g(p) \geq 0 \end{cases} \tag{3.5}$$

and

$$\partial/\partial p\, G_n(z, p) = 0 = \begin{cases} (c - fp)g'(p) - fg(p) + e\displaystyle\int_z^{\bar{\eta}} (\eta - z)\varphi(\eta)d\eta & \text{if } z + g(p) \geq 0 \\ [c - (f - e)p]g'(p) - (f - e)g(p) & \text{if } z + g(p) < 0. \end{cases} \tag{3.6}$$

The right-hand sides of (3.5) and (3.6) are both continuous in $z$ and $p$, except possibly at $z = -g(p)$.

In order to make the problem well-posed, we will present a set of seven assumptions which will be explained below. We will refer to these assumptions as Condition A. For any function $k(\cdot)$, let the notation $k'(a^+)$ denote the right-hand derivative at $a$.

Condition A:

(i) $c + h'(0^+) - \alpha c(1 - u) > 0$

(ii) $c - v'(0^+) - \alpha ct < 0$ if $z + g(p) < 0$

$c - v'(0^+) - \alpha ct - ec \leq 0$ if $z + g(p) \geq 0$

(iii) $fp > c$

(iv) Let $\hat{p}$ be the solution to $(c - fp)g'(p) - fg(p) = 0$

(v) $p_0 > \hat{p}$

(vi) $g''(p)(c - fp) - 2fg'(p) > 0$

(vii) $0 \leq e \leq f \leq 1$.

The first assumption is trivially satisfied in the stationary case. In the nonstationary case, $\alpha c$ is the next period's discounted ordering cost per unit. Assumption (i) then says that it is more costly to order and stock a unit this period for use in a later period than it is to order the unspoiled equivalent the next period. Assumption (ii) has a similar interpretation. It is less costly to produce a unit this period than to stock out and produce the backlogged equivalent the next period. Let us illustrate with a few examples. If backlogging is allowed and the revenue is collected $\lambda$ periods after the time the orders are placed with the firm, then $t = 1$ and $e = 0$. Assumption (ii) then becomes $c - v'(0^+) - \alpha c < 0$ for all values of $z + g(p)$. If this condition is violated, it is obvious that it is cheaper for the firm to backlog at least some units, collect the revenues and never satisfy the backorders. If excess demand is lost in each period, $t = 0$ and $e = f$. Furthermore, we are just interested in solutions which specify $z + g(p) \equiv y \geq 0$ in this case, since there are not units backlogged from a previous period ($x \geq 0$). Assumption (ii) then reduces to $(1 - e)c - v'(0^+) \leq 0$. If in addition $\lambda = 0$, then $e = 1$ and the condition becomes simply $v'(0^+) \geq 0$. Our assumptions mean that all the demand which is backlogged in a period will be met in the next period. It may also be helpful to note that even in the nonstationary case for

the deterministic problem obtained by setting $\eta \equiv 0$, assumptions (i) and (ii) imply that it is optimal to order an amount equal to the period's demand. There is no incentive to backlog or to keep an inventory.

Assumption (iii) expresses that the discounted price should be greater than the unit purchasing cost since there otherwise would be no desire for the firm to do business.

Item (iv) defines the price $\hat{p}$ which is that price which equates the marginal discounted revenue with respect to quantity with the marginal purchasing cost in a deterministic problem obtained by setting $\eta \equiv 0$.

Assumption (v) says that the largest price at which the demand is always positive should be greater than $\hat{p}$. We will later show that $\hat{p}$ is an upper bound on the optimal price in the stochastic problem. So the treatment is limited to problems where the price constraint will not be binding. If the price constraint were binding, an additive demand curve is clearly not a good description of the demand process.

Assumption (vi) implies that the solution to (iv) is unique.

Assumption (vii) will always be satisfied for all models of practical interest.

This discussion indicates that the optimal solution $(y_n^*, p_n^*, z_n^*)$ to the P-constrained problem has the properties $y_n^* > 0$, $p_n^* \in (c, p_0)$ and $z_n^* \in (\underline{\eta}, \bar{\eta})$. The following theorem will prove this.

THEOREM 1: If condition A is true, then the optimal solution to the P-constrained problem is given by $z_n^* \in (\underline{\eta}, \bar{\eta})$, $p_n^* \in (c, p_0)$ and $y_n^* > 0$.

PROOF: The theorem will be proved by induction. The first step is to show the validity of the theorem for $n = 1$. First note that since $g(p) + \underline{\eta} \geq 0$ for $p \leq p_0$, it follows that the condition $z + g(p) < 0$ implies $z < \underline{\eta}$. Then use (3.5) and assumptions (ii), (iii), and (vii) of condition A to show that $\partial/\partial z G_1(z,p) < 0$ for $z < \underline{\eta}$. From (3.5) and (i) of A it follows that $\partial/\partial z\, G_1(z, p) > 0$ for $z > \bar{\eta}$. This shows that $z_1^*$ is contained in the set $(\underline{\eta}, \bar{\eta})$. Since $z + g(p) < 0$ implies that $z < \underline{\eta}$, the analysis of (3.6) can be limited to the case $z + g(p) \geq 0$. We also have that

$$\sup_{z \in (\underline{\eta}, \bar{\eta})} \int_z^{\bar{\eta}} (\eta - z)\phi(\eta)\, d\eta = g(p_0) \tag{3.7}$$

and

$$\inf_{z \in (\underline{\eta}, \bar{\eta})} \int_z^{\bar{\eta}} (\eta - z)\phi(\eta)\, d\eta = 0. \tag{3.8}$$

Since $eg(p_0) < fg(c)$, it follows that

$$\partial/\partial p\, G_1(z, c)\,|_{z \in (\underline{\eta}, \bar{\eta})} \leq (c - fc)g'(c) - fg(c) + eg(p_0) < 0.$$

By using (vi) of A,

$$\partial/\partial p\, \{[c - fp]g'(p) - fg(p)\} = g''(p)[c - fp] - 2fg'(p) > 0. \tag{3.9}$$

Then apply (3.8), (3.9), and (iv) and (v) of A to find

$$\partial/\partial p\ G_1(z, p_0)\big|_{z\epsilon(\underline{\eta}, \bar{\eta})} \geq (c-fp_0)g'(p_0) - fg(p_0) > (c-f\hat{p})g'(\hat{p}) - fg(\hat{p}) = 0.$$

This shows that $p_1^*$ is contained in the set $(c, p_0)$.

Since $z_1^* > \underline{\eta}$, this implies that $y_1^* > \underline{\eta} + g(p_1^*) > \underline{\eta} + g(p_0) = 0$.

This proves the theorem for $n=1$.

Now assume that the theorem holds for the $n-1$ period problem. Since $y_{n-1}^* > 0$, it follows from (3.3) that if $z \leq \underline{\eta}$, then $\partial/\partial z\ F_{n-1}[t(z-\eta)] = -tc$. Also note that

$$\partial/\partial z\ F_{n-1}[(1-u)(z-\eta)] \geq -(1-u)c,$$

Then use similar arguments to those used for $n=1$ to complete the proof.

Theorem 1 shows that the price constraint is not binding, and for the purpose of finding $z_n^*$, $p_n^*$, $y_n^*$, the analysis of (3.5) and (3.6) can be limited to the case $z+g(p) \geq 0$.

Assumption (i) of A may be relaxed. Then there is a possibility that $z_n^* > \bar{\eta}$, in which case (3.5) and (3.6) can be solved separately and $p_n^* = \hat{p}$.

COROLLARY: Under the conditions of Theorem 1, $\hat{p}$ is an upper bound on the optimal price $p_n^*$ *to* the P-constrained problem. The upper bound is attained if $e=0$ or if $\eta=0$.

PROOF: Theorem 1 showed that the price constraint will not be binding for an optimal solution to the P-constrained problem. The optimal solution will have to satisfy (3.6). The right-hand side of (3.6) is continuous and differentiable with respect to $z$ and we obtain

$$\frac{dp}{dz} = \frac{e(1-\Phi(z))}{g''(p)(c-fp) - 2fg'(p)} \geq 0 \tag{3.10}$$

which shows that $p_n^*$ is a nondecreasing function of $z$. For $z \geq \bar{\eta}$ we find that the solution to (3.6) is $p=\hat{p}$. If $e=0$, or $\eta \equiv 0$, then the solution to (3.6) is $p=\hat{p}$ regardless of the value of $z$.

The economic meaning of the upper bound on $p_n^*$ follows directly from our discussion of assumption (iv) of A. So the optimal price $p_n^*$ to the P-constrained problem is less than or equal to that price which equates marginal discounted revenue with respect to quantity with marginal ordering cost in a deterministic problem obtained by setting $\eta \equiv 0$.

The next step is to investigate all possible solutions to (3.5) and (3.6). In the case $e>0$ define three new functions $Q(z)$, $N(z)$, and $T(z)$. $N(z)$ and $T(z)$ are those $p$-values which solve (3.6) and (3.5) respectively for a given value of $z$. Finally, let $Q(z)$ be the difference between $T(z)$ and $N(z)$. Both $T(z)$ and $N(z)$ are called single-valued functions.

$$Q(z) = T(z) - N(z) = \frac{c+L_n'(z)}{e(1-\Phi(z))} - N(z). \tag{3.11}$$

A necessary and sufficient condition for (3.5) and (3.6) to have a solution is that $Q(z)=0$ for some $z$.

If we can prove that (3.5) and (3.6) will yield a unique minimum to $G_n(z, p)$ s.t. $p \leq p_0$, then the optimal policy for the PP-constrained problem will resemble the single critical number policy in inventory

theory. The next theorem will prove uniqueness by presenting a condition which lacks immediate economic interpretation, but it will be shown later that this condition is satisfied for a variety of problems.

THEOREM 2: If the assumptions of Theorem 1 are true and if

(i) $L_n'(z)$ is differentiable for $z\epsilon(\underline{\eta}, \bar{\eta})$ except possibly on a set of measure zero,

(ii) $$\frac{\{L_n''(z)(1-\Phi(z))+\phi(z)(L_n'(z)+c)\}[g''(N(z))(c-fN(z))-2fg'(N(z))]}{e^2(1-\Phi(z))^3}$$

is strictly increasing in $z$ for $z\epsilon(\underline{\eta}, \bar{\eta})$ in the case $e>0$, and

$$L_n''(z)>0 \qquad \text{for } z\epsilon(\underline{\eta}, \bar{\eta}) \text{ in the case } e=0,$$

then (3.5) and (3.6) yield a unique minimum $z_n^*$, $p_n^*$ to the P-constrained problem.

PROOF: Case (a) $e>0$. Use assumption (ii) of A to establish that $Q(\underline{\eta})<0$. It follows easily that $Q(\bar{\eta})=\infty$. Let us examine the smallest $z>\underline{\eta}$ which satisfies $Q(z)=0$. $dQ(z)/dz$ must be nonnegative at this point. This derivative can be written.

(3.12) $$\frac{dQ(z)}{dz}=\frac{L_n''(z)+eT(z)\phi(z)}{e(1-\Phi(z))}-\frac{e(1-\Phi(z))}{g''(N(z))(c-fN(z))-2fg'(N(z))}.$$

If (3.12) is strictly positive, then there will be a unique solution to $Q(z)=0$ for $z\epsilon(\underline{\eta}, \bar{\eta})$. By using assumption (vi) of A and the definition of $T(z)$, it follows that $dQ(z)/dz$ will be strictly positive if condition (ii) of this theorem is met. Consequently, $z_n^*$ and $p_n^*$ are unique. The Hessian of $G_n(z, p)$ is

(3.13) $$H=\partial^2/\partial z^2\, G_n(z,p)\;\partial^2/\partial p^2\, G_n(z,p)-[\partial^2/\partial z\partial p\, G_n(z,p)]^2$$

$$=[L_n''(z)+ep\phi(z)][g''(p)[c-fp]-2fg'(p)]-e^2(1-\Phi(z))^2.$$

For the values $z_n^*$, $p_n^*$, $T(z_n^*)=N(z_n^*)=p_n^*$. By comparing (3.12) and (3.13) it follows that the Hessian is positive, and a global minimum is attained for the P-constrained problem at the point $z_n^*$, $p_n^*$.

Case (b) $e=0$. (3.6) yields the unique solution $p=\hat{p}$. Condition (ii) shows that there is likewise a unique solution to (3.5) since $\partial^2 G_n(z,p)/\partial z^2=L_n''(z)>0$. The Hessian is positive.

Corresponding to $z_n^*$ and $p_n^*$ there is a unique optimal value $y_n^*$ equal to $z_n^*+g(p_n^*)$. For $x\leq y_n^*$, the PP-constrained problem is equivalent to the P-constrained problem. It is optimal to order up to $y_n^*$ and charge the price $p_n^*$. The next theorem shows the complete optimal policy for the PP-constrained problem. Two additional conditions are needed to show the uniqueness of this policy. It will be shown later that these conditions are met for a large variety of cases.

Rewrite $G_n(z, p)$ as $\bar{G}_n(y, p)$. From (3.2) we find that

$$\bar{G}_n(y,p)=cy+L_n(z)+ep\int_z^{\bar{\eta}}(\eta-z)\phi(\eta)d\eta-fpg(p).$$

Furthermore,

$$(3.14)\quad \partial/\partial_p \bar{G}_n(y, p) = 0 = [-L_n'(z) + ep(1-\Phi(z)) - fp]g'(p) - fg(p) + e\int_z^{\bar{\eta}} (\eta - z)\phi(\eta)d\eta.$$

THEOREM 3: Under the conditions of Theorem 2 and if

(i) $$-g''(p)[L_n'(z) - ep(1-\Phi(z)) + fp] + g'(p)[e(1-\Phi(z)) - 2f] > 0 \text{ and}$$

(ii) $$g'(p)^2[L_n''(z) + ep\phi(z)] - g''(p)[L_n'(z) - ep(1-\Phi(z)) + fp] + g'(p)[2e(1-\Phi(z)) - 2f] > 0,$$

then the optimal solution to the PP-constrained problem is given by:

If $x \leq y_n^*$, order up to $y_n^*$ and set the price equal to $p_n^*$. $z_n^*$ and $p_n^*$ can be found by solving (3.5) and (3.6). Then set $y_n^* = z_n^* + g(p_n^*)$.

If $x > y_n^*$, do not order. Set the price equal to $p_n^*(x)$ which can be found by solving (3.14) where $z$ is replaced by $x - g(p)$.

PROOF: The analysis of the P-constrained problem proves the theorem for $x \leq y_n^*$, if $x > y_n^*$, the solution will be a boundary solution since the P-constrained problem has a unique local minimum at $y = y_n^*$. We will show that the constraint $p \leq p_0$ is not binding for $x \geq y_n^*$ which in turn implies that $z \geq x - g(p)$ has to be binding and hence the optimal inventory level is equal to $x$. Use (3.14) to define a new function $S(z)$ such that for a given value of $z$, $S(z)$ will be that $p$-value which solves (3.14). $S(z)$ is a single valued function, since for a given $z$ the partial derivative of (3.14) with respect to $p$ is positive under condition (i). Use (3.14), assumptions (i), (iv), (v), and (vi) of A to show that $S(\bar{\eta}) \leq N(\bar{\eta}) = \hat{p} < p_0$. Since $N(z) < p_0$ for all $z$ and since there is a unique $z_n^*(\underline{\eta}, \bar{\eta})$ which makes $T(z) = N(z) = S(z)$, it follows that $S(z) < N(z) < p_0$ for $z > z_n^*$. The price constraint is not binding for $x > y_n^*$. The optimal price $p_n^*(x)$ can be found by solving (3.14), where $z$ is replaced by $x - g(p)$. This price is unique because $\partial^2/\partial p^2 \bar{G}_n(x, p) > 0$ according to assumption (ii).

Our intuition tells us that under reasonable assumptions $p_n^*(x)$ should be a decreasing function of $x$. In order to establish conditions under which this is the case, we need some information regarding $S(z)$.

From (3.14) and the definition of $S(z)$:

$$(3.15)\quad \frac{dS(z)}{dz} = \frac{L_n''(z)g'(p) + e(1-\Phi(z)) + epg'(p)\phi(z)}{-2fg'(p) + g''(p)[-L_n'(z) - fp + ep(1-\Phi(z))] + e(1-\Phi(z))g'(p)}.$$

LEMMA 1: If (a); condition (i) of Theorem 3 is true, and (b); $L_n''(z) \geq 0$ for $z \geq \bar{\eta}$, and (c); in the case $e > 0$,

$$\frac{L_n''(z)g'(p) + epg'(p)\phi(z)}{1-\Phi(z)} + e$$

changes sign at most once (from positive to negative) as $z$ is increased from $\underline{\eta}$ to $\bar{\eta}$, then (3.15) changes sign at most once (from positive to negative) for $z > \underline{\eta}$.

PROOF: By assumption (a) the denominator of (3.15) is positive. Then rewrite the numerator as

$$\left[\frac{L_n''(z)g'(p)+epg'(p)\phi(z)}{1-\Phi(z)}+e\right](1-\Phi(z)),$$

which will change sign at most once (from positive to negative) for $z\epsilon(\underline{\eta}, \bar{\eta})$ under condition (c) of this lemma. Similarly, the condition $L_n''(z) \geq 0$ proves the lemma for $z \geq \bar{\eta}$.

We are now able to make inferences about $S(z)$. If $e=0$, the assumptions of Lemma 1 in conjunction with requirement (ii) in Theorem 2 state that $L_n''(z) \geq 0$ for $z > \underline{\eta}$; thus it is easy to see that (3.15) is decreasing for $z > z_n^*$ in this case. If $e > 0$, we need the results of the next lemma.

LEMMA 2: Let $\tilde{p}$ be the solution to

$$g'(\tilde{p})e\tilde{p}+L_n'(\underline{\eta})(-g'(\tilde{p}))-e\underline{\eta}-fg(\tilde{p})-f\tilde{p}g'(\tilde{p})=0.$$

If $e > 0$, and the conditions of Lemma 1 are satisfied, and if $\tilde{p} \geq \hat{p}$, then $S(z)$ is a decreasing function of $z$ for $z \geq z_n^*$.

PROOF: The analysis in Theorem 3 and the derivative given by (3.10) establish that $S(z_n^*) = N(z_n^*) < N(\bar{\eta}) = \hat{p}$. Furthermore, $S(\underline{\eta}) = \tilde{p} \geq \hat{p}$, and since $dS(z)/dz$ changes sign at most once (from positive to negative), it follows that for $z \geq z_n^*$, $S(z)$ is a decreasing function of $z$.

We can now make inferences about $p_n^*(x)$.

THEOREM 4: Under the conditions of Lemma 2, the optimal price in the PP-constrained problem is a decreasing function of starting inventory.

PROOF: If $x \leq y_n^*$, then $p_n^*(x) = p_n^*$.

If $x > y_n^*$, then we find by using the relationship $z = x - g(p_n^*(x))$ that

$$\frac{dp_n^*(x)}{dx} = \frac{dS(z)/dz}{1+g'(p_n^*(x))\ dS(z)/dz}. \tag{3.16}$$

According to Lemma 2, the denominator is positive while the numerator is nonpositive. $p_n^*(x)$ is a decreasing function of $x$.

We will later need properties of $F_n(x)$. The next theorem will show that $F_n(x)$ is convex if $dp_n^*(x)/dx$ is nonpositive.

THEOREM 5: Under the conditions of Theorems 1, 3, and 4, $F_n(x)$ is convex.

PROOF: From Theorem 3 it is easy to see that the theorem holds for $x \leq y_n^*$ since

$$d^2/dx^2\ [\bar{G}_n(y_n^*, p_n^*)-cx]=0.$$

If $x > y_n^*$, then use $\partial/\partial p\ \bar{G}_n(x, p_n^*(x)) = 0$ to obtain

$$d^2/dx^2\ \bar{G}_n(x, p_N^*(x)) = [1-g'(p_n^*(x))\ dp_n^*(x)/dx]\ [L_n''(z) + ep_n^*(x)\phi(z)] - e\ dp_n^*(x)/dx\ (1-\Phi(z)). \tag{3.17}$$

By applying (3.16) it follows that $1-g'(p_n^*(x))\ dp_n^*(x)/dx$ can be written $(1+g'(p_n^*(x))dS(z)/dz)^{-1}$, which is positive for $x \geq y_n^*$ by Lemma 2. Use assumptions (i), (ii), (iii), and (vii) of A and the convexity of $h(\cdot)$ and $v(\cdot)$ to show that $L_n''(z)+ep_n^*(x)\phi(z)$ is positive for $n=1$. It then follows that $d^2/dx^2G_1|x, p_1^*(x))$ is positive, and $F_1(x)$ is convex. Now assume that $F_{n-1}(x)$ is convex. Note that $F'_{n-1}(x)=-c$ for $x \leq 0$ and present similar arguments to those above to show that $F_n(x)$ is convex.

## 4. SPECIAL CASES OF THE GENERAL $n$-PERIOD PROBLEM

The treatment so far has been kept quite general. But we have not achieved generality without imposing conditions which lack intuitive appeal and do not lend themselves to an easy interpretation. The restrictiveness of these assumptions is also not clear. This section will alleviate these problems as we will present more specific cases for which the conditions mentioned above are satisfied.

### Linear Expected Demand Curve and Linear Stockout Costs

In this section we will assume that in each period the riskless demand curve is linear and also that the stockout cost is linear, in particular $g(p)=a+bp$ and $v(\eta-z)=v\cdot(\eta-z)$. Initially we will make some comments regarding probability distributions with increasing failure rate (IFR) and Polya frequency functions of the second order ($PF_2$).

A distribution is defined as IFR if $\phi(\eta)/[1-\Phi(\eta)]$ is nondecreasing in $\eta$. A distribution is called $PF_2$ if $\phi(\eta)/[\Phi(\eta+s)-\Phi(\eta)]$ is nondecreasing in $\eta$ for all real $s$. When $s\to\infty$ the condition reduces to the requirement that $\phi(\eta)/[1-\Phi(\eta)]$ is nondecreasing in $\eta$. It follows that $PF_2$ is a subclass of IFR. Some common distributions which are $PF_2$ are the uniform, the normal and the truncated normal. Among the IFR group we find, in addition to the ones just mentioned, the noncentral $T$ and $F$ densities. The Gamma distribution and the Weibull distribution with parameter $\lambda \geq 1$ have IFR. They coincide with the exponential for $\lambda=1$. The exponential has constant failure rate and belongs to the IFR group. A discussion concerning IFR and $PF_2$ can be found in Barlow and Proschan [2]. We will now prove an additional property of $PF_2$ densities. According to Barlow and Proschan [2] a density which is $PF_2$ has the property that $\phi(\eta-s)/\phi(\eta)$ is monotone nondecreasing in $\eta$ for all finite $s \geq 0$. We can then prove the following lemma.

LEMMA 3: If $\phi(\eta)$ is $PF_2$, then $\phi(\eta-s)/[1-\Phi(\eta)]$ is nondecreasing in $\eta$ for all finite $s \geq 0$.

PROOF: $PF_2$ implies that $\phi(\eta-s)/\phi(\eta)$ is nondecreasing in $\eta$ for all finite $s \geq 0$. Since $PF_2$ is a subset of IFR, we also have for $PF_2$ densities that $\phi(\eta)/[1-\Phi(\eta)]$ is nondecreasing in $\eta$. Consequently $\dfrac{\phi(\eta-s)}{\phi(\eta)}\cdot\dfrac{\phi(\eta)}{1-\Phi(\eta)}=\dfrac{\phi(\eta-s)}{1-\Phi(\eta)}$ is nondecreasing in $\eta$ for all finite $s \geq 0$. This property will help us prove the next lemma.

LEMMA 4: If $\phi(\eta)$ is $PF_2$ and $T(z)$ is a convex function, then

$$\frac{\int_{\eta}^{z} T''(z-\eta)\phi(\eta)d\eta}{1-\Phi(z)}$$

is nondecreasing in $z$.

PROOF: Set $s=z-\eta$ and define $M(z)$ as

$$M(z)=\frac{\int_{\underline{\eta}}^{z} T''(z-\eta)\phi(\eta)d\eta}{1-\Phi(z)}=\frac{\int_{\neq 0}^{z-\underline{\eta}} T''(s)\phi(z-s)ds}{1-\Phi(z)}.$$

Let $z_2 >_1 z_1$. Then

$$M(z_2)-M(z_1)=\int_0^{z_1-\underline{\eta}} T''(s)\left\{\frac{\phi(z_2-s)}{1-\Phi(z_2)}-\frac{\phi(z_1-s)}{1-\Phi(z_1)}\right\}ds+\int_{z_1-\underline{\eta}}^{z_2-\underline{\eta}} T''(s)\frac{\phi(z_1-s)}{1-\Phi(z_2)}\,ds.$$

Since $T''(s)$ is nonnegative due to the convexity of $T(s)$, it follows from Lemma 3 that $M(z_2)-M(z_1)$ is nonnegative.

We will now show that when the expected demand curve and stockout costs are linear, Theorems 2 and 3 will be satisfied automatically.

THEOREM 6: Under the conditions of Theorem 1 and if the expected demand curve and the stockout cost functions are linear and $\phi(\eta)$ is $PF_2$, then the solution to the PP-constrained problem is given by Theorem 3 and the optimal price is a decreasing function of starting inventory.

We will give the necessary ingredients to complete the proof by induction.

PROOF: Initially we want to show that $F_n(x)$ is convex under the conditions of this theorem. In order to show convexity we need to establish that $dp_n^*(x)/dx$ is negative which means that the conditions in Lemmas 1 and 2 must be satisfied. Since $y_{n-1}^* > 0$ implies that $F'_{n-1}(x)=-c$ for $x \leq 0$, the condition in Lemma 1 (for $e>0$) reduces to:

$$\frac{b\Big[[h'(0^+)+v-\alpha c(1-u-t)]\phi(z)+\alpha\int_{\underline{\eta}}^{z} F''_{n-1}[(1-u)(z-\eta)]\phi(\eta)d\eta}{1-\Phi(z)}$$

$$\frac{+\int_{\underline{\eta}}^{z} h''(z-\eta)\phi(\eta)d\eta\Big]+epb\phi(z)}{}+e.$$

The expression above is allowed to change sign at most once (from positive to negative) as $z$ is increased from $\underline{\eta}$ to $\bar{\eta}$. We will first show that this is true for $n=1$ and under the condition that $F_{n-1}(\cdot)$ is convex, it also holds for $n \geq 2$. The proof will be by contradiction. First of all recollect that corresponding values of $z$ and $p$ in (4.1) will jointly have to satisfy (3.14). We used the symbol $S(z)$ for these $p$-values and the derivative of $p$ with respect to $z$ is given in (3.15). Assume then that the derivative (3.15) is zero for $z=z^1$. Also assume that there exists a $z > z^1$ for which the derivative is positive. By using Lemma 4 these two assumptions lead to a contradiction in the case $n=1$ since the increase in $p$ and the increase in $z$ will decrease (4.1). Similar arguments in the case $n \geq 2$ show that (4.1) will decrease if $F_{n-1}(\cdot)$ is convex. We then have that the derivative is zero at $z=z^1$. According to the arguments above and the discussion in Lemma 1 an increase in $z$ will drive the numerator and consequently the derivative negative under the above assumptions. We conclude that condition c) in Lemma 1 is satisfied for $z\epsilon(\underline{\eta}, \bar{\eta})$. For $z>\bar{\eta}$ it is sufficient to show that $L_n''(z)\geq 0$. By using conditions (i) and (ii) of A this follows readily in the case $n=1$ and also for the case $n \geq 2$ if $F_{n-1}(\cdot)$ is convex.

Finally we find that the condition in Lemma 2 simplifies to

$$\tilde{p}=\frac{-ebp_0-ea+fa-b[-v-\alpha tc]}{-2fb+eb}\geq\frac{fa-cb}{-2fb}=\hat{p}.$$

This inequality will hold if

$$p_0 \geq \frac{-v-\alpha ct+c}{e}+\frac{-cb-fa}{2fb}.$$

Use assumption (ii) and (v) of A show that the inequality is met.

The arguments above show that Theorem 4 is applicable. $dp_n^*(x)/dx$ is negative for $n=1$ and will be negative for $n \geq 2$ if $F_{n-1}(\cdot)$ is convex. According to Theorem 5, $F_n(x)$ will be convex if $dp_n^*(x)/dx \leq 0$. After completing the check below of the remaining conditions, it follows that $F_n(x)$ is convex.

The linearity of the expected demand curve and the stockout cost function allows condition (ii) of Theorem 2 after some manipulation to be written

$$(4.2)\quad -2fb\,\frac{\phi(z)}{e^2(1-\Phi(z))^3}P_n(z)-2fb\,\frac{\alpha\int_{\underline{\eta}}^{z}F_{n-1}''[(1-u)(z-\eta)]\phi(\eta)d\eta+\int_{\underline{\eta}}^{z}h''(z-\eta)\phi(\eta)d\eta}{e^2(1-\Phi(z))^2}$$

where

$$P_n(z)=[h'(0^+)-\alpha c(1-u)](1-\Phi(z))+c+\int_{\underline{\eta}}^{z}h'(z-\eta)\phi(\eta)d\eta+\alpha\int_{\underline{\eta}}^{z}F_{n-1}'[(1-u)(z-\eta)]\phi(\eta)d\eta.$$

We find that $dP_n(z)/dz$ is positive. Since $\phi(\eta)$ is $PF_2$ which is a subset of IFR, then $\phi(z)/[1-\Phi(z)]^3$ is strictly increasing in $z$. By Lemma 4.

$$\frac{\int_{\underline{\eta}}^{z}h''(z-\eta)\phi(\eta)d\eta+\alpha\int_{z}^{\bar{\eta}}F_{n-1}''[(1-u)(z-\eta)]\phi(\eta)d\eta}{[1-\Phi(z)]^2}$$

is also strictly increasing in $z$. We conclude that (4.2) is strictly increasing in $z$ for $z\epsilon(\underline{\eta}, \bar{\eta})$ which satisfies condition (ii) of Theorem 2 for the case $e>0$. If $e=0$, we have from Theorem 2 that it is sufficient to show that $L_n''(z)>0$ for $z\epsilon(\underline{\eta}, \bar{\eta})$. This is easily verified by the use of assumptions (i) and (ii) of A. The conditions of Theorem 3 are likewise easily checked to hold.

**Backlogging With Revenue Collection λ Periods After the Time the Orders Are Placed With the Firm and a Concave Demand Curve**

In this case, $e=0$ and $g''(p) \leq 0$.

THEOREM 7: If $e=0$ and $g''(p) \leq 0$, then the solution to the PP-constrained problem is given by Theorem 3 and the optimal price is a decreasing function of starting inventory.

PROOF: The approach is similar to that in Theorem 6. It is easy to show by induction and with the help of assumptions (i) and (ii) of A that $L_n''(z)>0$ for $z>\underline{\eta}$. This satisfies Theorem 2 and implies that $dp_n^*(x)/dx$ is negative and $F_n(x)$ is convex. Condition (i) of Theorem 3 reduces to $g''(p)(fp+L_n'(z))+2fg'(p)<0$. We have that $L_n'(z)>h'(0^+)\Phi(z)-v'(0^+)(1-\Phi(z))-\alpha c(1-u)\Phi(z)-\alpha ct(1-\Phi(z))>-c\Phi(z)-c(1-\Phi(z))=-c$. It then follows easily that both conditions in Theorem 3 are met.

**Discussion**

When the revenue collection is dependent on the time the customers place their orders with the firm, an important distinction exists between the pure backlogging case ($e=0$) and the case where some or all excess demand is lost ($e>0$). When backlogging is allowed it is sufficient to assume linear ordering costs, convex holding and stockout costs, a concave expected demand curve and any probability distribution with a finite mean for the policy in Theorem 3 to be optimal. When only partial backlogging is allowed or when excess demand is lost, it is sufficient to assume linear ordering and stockout costs, convex holding costs, a linear expected demand curve and a $PF_2$ distribution.

The optimal values $z_n^*$ and $p_n^*$ are found by solving (3.5) and (3.6). For $e=0$ these two equations can be solved separately, and the optimal price is equal to that price which equates marginal expected discounted revenue with respect to quantity with the marginal ordering cost in the period. The amount of risk as expressed by the standard deviation of $\eta$ has no effect on the optimal price. Since excess demand is backlogged, all the demand in a period will ultimately be satisfied at the price selected. There is no opportunity cost associated with stockouts and consequently no incentive to set the price lower than the optimal price in a deterministic model obtained by setting $\eta=0$. In the case $e>0$, there are opportunity costs associated with lost sales and the optimal price is depressed.

When the revenue collection is dependent on the time the orders are satisfied, the above mentioned distinction between backlogging and lost sales vanishes. In both cases we have that $e>0$ and the same sufficiency conditions as above hold. There is an opportunity cost of backlogging as well as of lost sales and the optimal price is depressed.

## 5. CONCLUSION

The general $n$-period problem has been solved and the sufficiency conditions have been shown to hold for a variety of practical cases. Without difficulty all the results can be generalized to the nonstationary case.

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# INTERSECTION CUTS FROM OUTER POLARS OF TRUNCATED CUBES*

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**ABSTRACT**

A truncated cube, by which we mean the convex hull of a subset of the vertices of the unit cube, has an outer polar whose facets are a subset of the facets of the octahedron (the outer polar of the cube). We discuss procedures for generating valid truncations of the cube from the problem constraints, in the case of 0–1 integer programs, and for intersecting the halflines defined by the constraints that are tight for a basic solution to the linear program, with successive facets of the outer polars of these truncated cubes. The cutting planes obtained in this way are compared to other cuts.

## 1. INTRODUCTION

Consider the problem

$$\text{(P)} \qquad \max\{cx \mid x \in X, \quad x \text{ integer}\},$$

where

$$X = \{x \in R^n \mid Ax \leq b, \quad 0 \leq x \leq e\}$$

and where $A$ is $m \times n$, $b$ is $m \times 1$, while $e = (1, \ldots, 1) \in R^n$. With $Ax \leq b$ stated as $Ax + z = b$, let $t = (x, z)$, let $\bar{t} = (\bar{x}, \bar{z})$ be an optimal solution to the linear program $\max\{cx \mid x \in X\}$, and $x = \bar{x} - \sum_{j \in J} \bar{a}_j t_j$, where $J$ is the index set for the nonbasic variables at the linear programming optimum.

For any set $T \subset R^n$ let the boundary, the interior, the convex hull, the affine hull, and the set of vertices of $T$ be denoted by bd $T$, int $T$, conv $T$, aff $T$ and vert $T$, respectively.

The recently developed convex analysis approach to integer programming uses as its starting point the following basic result (see [1], [8], [2] for various forms and proofs of it).

THEOREM 1.1: Let $T \subset R^n$ be any closed convex set whose interior contains $\bar{x}$, but no integer point of $X$, and let $\bar{\lambda}_j$ be the value of $\lambda_j$ for which the halfline $\xi_j = \{x \mid x = \bar{x} - \bar{a}_j \lambda_j, \ \lambda_j \geq 0\}$ intersects the boundary of $T$ (with $\bar{\lambda}_j = +\infty$ if no such intersection exists). Then the inequality

$$\sum_{j \in J} \bar{\lambda}_j^{-1} t_j \geq 1 \tag{1.1}$$

defined by the $n$ intersection points $\xi_j \cap \text{bd } T = \bar{x} - \bar{a}_j \bar{\lambda}_j$, $j \in J$, is a valid cut (i.e., does not cut off any integer points of $X$).

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Cuts of the type defined in Theorem 1.1, called intersection (or convexity) cuts, have been generated from the ball [1] and the regular octahedron [6] circumscribing the unit cube $K=\{x\epsilon R^n|0\leqslant x\leqslant e\}$, as well as from other convex sets [10], [8], [7], [2]. In [2], the family of outer polar cuts was introduced. The outer polar of a set $S\subset R^n$ is defined as

$$S^*=S^0(n/4)=\{y|(y-1/2\,e)\,(x-1/2\,e)\leqslant n/4,\qquad \forall\, x\epsilon S\}.$$

i.e., the (translated) scaled polar of $S$, with scaling factor $n/4$. For any set $S$ such that

$$S\supseteq \text{conv}\,\{x\epsilon X|x \text{ integer}\},$$

it can be shown [2] that int $S^*$ contains $\bar{x}$, but no integer point of $X$ (by the choice of the scaling factor $n/4$, all vertices of $K$ contained in $X$ are also contained in the boundary of $S^*$). The outer polar $S^*$ can thus be used to generate intersection cuts. The smaller the set $S$, the larger $S^*$ and the stronger the cuts tend to be. This family of cuts contains the cut from the octahedron $K^*$ (the outer polar of $K$), which is computationally cheap, as well as the one from $X^*$, the outer polar of the feasible set, which is computationally more expensive, but uniformly stronger than the cut from $K^*$. In this paper we generate intersection cuts from the outer polars of sets of the form

$$Q=\left\{x\epsilon R^n|\sum_{i\epsilon N_k}x_i\leqslant r_k,\qquad k=1,\ldots,s,\qquad 0\leqslant x\leqslant e\right\},$$

which we call truncated cubes. Here $\{N_k\}_{k=1}^{k=s+1}$ is a partition of $N$ and $r_k$ is an integer satisfying $1\leqslant r_k\leqslant |N_k|-1$. We will denote by $N_{s+1}=N-\bigcup_{k=1}^{s}N_k$ the index set for those $x_i$ not present in any of the constraints

$$\sum_{i\epsilon N_k}x_i\leqslant r_k,\qquad k=1,\ldots,s. \tag{1.2}$$

The property that distinguishes sets of the form $Q$ from other subpolytopes of $K$ is that vert $Q\subset$ vert $K$, i.e., each set $Q$ is the convex hull of a subset of the vertices of $K$.

The constraints (1.2) are canonical cuts [5] implied by the inequalities defining $X$. The basic idea of our approach is that by generating for each halfline $\xi_j$ a set of properly chosen disjoint canonical cuts (i.e., such that no variable appears in more than one cut), we create a set $Q_j$ which is a cube "truncated in the direction of $\xi_j$." The outer polar $Q_j^*$ of $Q_j$ is then of such simple form that intersecting $\xi_j$ with boundary of $Q_j^*$, and then with extended facets of $Q_j^*$, is computationally cheap.

Of course, when some of the constraints of $X$ are themselves of the form (1.2), as is the case with multiple choice constraints, they can be directly used to generate the sets $Q_j$.

The inequalities (1.2) actually represent a more general type of (canonical) constraints. For instance, any polyhedron defined by constraints of the type

$$\sum_{i\epsilon N_k^+}x_i-\sum_{i\epsilon N_k^-}x_i\leqslant r_k,\qquad \text{where } -|N_k^-|\leqslant r_k\leqslant |N_k^+|-1, \tag{1.3}$$

can be brought to the form of $Q$ by complementing the variables $x_i$, $i \in N_k^-$. Further, a truncated cube defined by equality rather than inequality constraints can be replaced by an equivalent truncated cube of the type $Q$, by a device to be discussed below.

It can be shown (see [2]) that:

(i) $$K^* = \{y|(y - 1/2\,e)\ (x - 1/2\,e) \leqslant n/4, \quad \forall\, x \in K\}$$

$$= \{y|(y - 1/2\,e)\ x \leqslant 1/2\,ey, \quad \forall\, x \in \text{vert } K\}$$

$$= \{y|\ |y - 1/2\,e| \leqslant n/2\}$$

(ii) $$Q^* = \{y|(y - 1/2e)(x - 1/2\,e) \leqslant n/4, \quad \forall\, x \in Q\}$$

$$= \{y|(y - 1/2e)x \leqslant 1/2ey, \quad \forall\, x \in \text{vert } Q\}$$

(iii) $$Q \subset K \subset K^* \subset Q^*$$

(iv) $$\text{bd } K^* = \{y|\max\{(y - 1/2\,e)x|x \in K\} = 1/2\,ey\}$$

(v) $$\text{bd } Q^* = \{y|\max\{(y - 1/2\,e)x|x \in Q\} = 1/2\,ey\}$$

(vi) Finding the intersection of a halfline $\xi = \{x|x = \bar{x} - \bar{a}\lambda, \lambda \geqslant 0\}$ with bd $Q^*$ amounts to finding $\lambda$ such that

(PLP) $$\max\left\{\sum_{i \in N} p_i(\lambda)x_i | x \in Q\right\} = p_0(\lambda)$$

where $$p_i(\lambda) = \bar{x}_i - 1/2 - \bar{a}_i\lambda \qquad \text{and} \qquad p_0(\lambda) = 1/2 \sum_{i \in N} (\bar{x}_i - \bar{a}_i\lambda)\,.$$

(Here and wherever it is possible without creating confusion, the subscripts on $\xi$, $\bar{a}$, $\lambda$ and $Q^*$ are omitted.)

(vii) There is a one-to-one mapping, $\varphi$, between $x \in$ vert $K$ ($x \in$ vert $Q$) and facets, $\varphi(x)$, of $K^*$ (of $Q^*$). Also, $x \in \varphi(X)$ for all $x \in$ vert $K$ ($x \in$ vert $Q$), i.e., every vertex of the cube (truncated cube) lies on the associated facet of $K^*$ (of $Q^*$).

If the halflines $\xi$ are intersected not only with bd $K^*$ (bd $Q^*$), but also with subsequent (extended) facets of $K^*$ (of $Q^*$), stronger inequalities are obtained, which are valid cuts provided that the vertex $v$ of $K$ (of $Q$) contained in each facet $\varphi(v)$ of $K^*$ (of $Q^*$) that is intersected, is checked for feasibility.

Since vert $Q \subset$ vert $K$, it follows that $\{\varphi(\text{vert } Q)\} \subset \{\varphi(\text{vert } K)\}$. The extended facets of $K^*$ containing vertices cut off by constraints (1.2) are not facets of $Q^*$. Since constraints (1.2) cut off $\eta$ vertices of $K$, $\eta$ facets of $K^*$ are missing (i.e., are not facets of $Q^*$), where

$$\eta = \sum_{k=1}^{s} \left[ \sum_{h=1}^{|N_k| - r_k} \binom{|N_k|}{r_k + h} \right] 2^{n - |N_k|}.$$

Consequently, the halfline $\xi$ intersects at most as many, but usually fewer, extended facets of $Q^*$,

than of $K^*$. Cuts derived from intersecting the extended facets of $Q^*$ are uniformly deeper than those derived from intersecting an equal number of extended facets of $K^*$.

As mentioned before, a truncated cube defined by equality constraints, $\sum_{i\in N_k} x_i = r_k$, can be replaced by an equivalent truncated cube of the above type $Q$. This can be achieved by introducing an artificial variable $z_k$ with a cost $M_k$ in the objective function of (PLP). Since (PLP) always has a feasible solution with $z_k = 0$, for $M_k$ sufficiently large, an optimal solution will always have $z_k = 0$. Eliminating $z_k$ from the objective function by using $z_k = r_k - \sum_{i\in N_k} x_i$ therefore yields a truncated cube $Q$ defined by inequalities having the same optimal solution set as the initial equality-constrained truncated cube.

The polyhedron $Q$ introduced above is a truncated $n$-dimensional cube. However, intersection cuts can also be generated from $n$-dimensional extensions of the outer polars of lower dimensional truncated cubes. Let $M \subseteq N$ such that $\bigcup_{k=1}^{s} N_k \subseteq M$ and $|M| = m$. Let $R^m$ be an $m$-dimensional subspace of $R^n$, and let $x = (x^m, x^{n-m}) \in R^m \times R^{n-m}$, where $x^m$ and $x^{n-m}$ are indexed by $M$ and $N - m$ respectively. Consider now the $m$-dimensional truncated cube

$$Q_m = \left\{ x^m \in R^m \,\middle|\, \sum_{i\in N_k} x_i \leq r_k, \qquad i = 1, \ldots, s, \qquad 0 \leq x^m \leq e^m \right\},$$

and its outer polar

$$Q_m^* = \{ y^m \in R^m \,|\, (x^m - 1/2\, e^m)\,(y^m - 1/2\, e^m) \leq m/4, \qquad \forall x^m \in Q_m \}.$$

The $n$-dimensional extension of $Q_m^*$ is the set

$$Q_{m(n)}^* = Q_m^* + R^{n-m},$$

which has the properties required for yielding valid intersection cuts.

Denoting

$$Q = \left\{ x \in R^n \,\middle|\, \sum_{i\in N_k} x_i \leq r_k, \qquad k = 1, \ldots, s, \qquad 0 \leq x \leq e \right\},$$

it can be easily shown that $Q_{m(n)}^*$ is the outer polar of

$$Q_{m(n)} = \{ x \in Q \,|\, x^{n-m} = 1/2\, e^{n-m} \}$$

Hence $\xi \cap \operatorname{bd} Q_{m(n)}^*$ can be obtained by finding the $\lambda$ which solves the problem:

$$\max \left\{ \sum_{i\in M} p_i(\lambda) x_i \,\middle|\, x^m \in Q_m \right\} = p_0(\lambda) - 1/2 \sum_{i\in N-M} p_i(\lambda).$$

Since each facet of $Q_{m(n)}^*$ contains $2^{n-m}$ vertices of $K$, a facet can be dropped only after checking

$2^{n-m}$ vertices for feasibility. For this reason, intersection cuts from outer polars of higher dimensional truncated cubes seem more desirable.

The algorithms discussed in the next three sections will be stated in terms of $n$-dimensional truncated cubes; but in view of the above, everything carries through to lower-dimensional truncated cubes.

The intersection $\xi \cap \text{bd } Q^*$ can be found by solving (PLP). This amounts to finding the zero of a convex piecewise-linear function of a very simple form. The next section (2) describes an algorithm for doing this. Section 3 then gives an algorithm for obtaining the extended facets of $Q^*$ in the order in which they are intersected by $\xi$. In Section 4 possible strategies are discussed for generating appropriate truncated cubes, and an algorithm is described for generating truncated cubes having certain desirable properties. Finally, the last section (5) relates some of the cuts described in this paper to Glover's cut for 0–1 problems with a multiple choice constraint.

## 2. ALGORITHM 1 (FOR FINDING $\xi \cap \text{bd } Q^*$)

The function

$$z(\lambda) = \max\left\{\sum_{i\in N} p_i(\lambda)x_i \mid x\in Q\right\}$$

is convex and piecewise-linear. Hence, $F(\lambda)=z(\lambda)-p_0(\lambda)$ is also a convex piecewise linear function of $\lambda$. In fact, if $\xi \cap \text{bd } Q^*=\bar{x}-\bar{a}\bar{\lambda}$, then $F(\lambda)<0$ for $0\leq\lambda<\bar{\lambda}$; $F(\lambda)=0$ for $\lambda=\bar{\lambda}$; and $F(\lambda)>0$ for $\lambda>\bar{\lambda}$ (see [2]). If $F(\lambda)<0$ for all $\lambda\geq 0$ then $\xi \cap \text{bd } Q^*=\emptyset$.

For any $\lambda$, let

$$N_k^+(\lambda)=\{i\in N_k \mid p_i(\lambda)>0\} \text{ and } N_k^-(\lambda)=\{i\in N_k \mid p_i(\lambda)\leq 0\} \text{ for } k=1,\ldots,s+1;$$

if we define an ordering on $N_k^+$, namely

$$N_k^+(\lambda)=\{k_1,\ldots,k_{\mu_k}\} \quad \text{where } p_{k_i}(\lambda)\geq p_{k_j}(\lambda) \quad \text{if } 1\leq i<j\leq\mu_k \quad \text{for } k=1,\ldots,s;$$

and denote

$$\sigma_k=\min\{r_k,\mu_k\} \quad \text{for } k=1,\ldots,s,$$

then the set of the $\sigma_k$ largest coefficients $p_{k_i}(\lambda)$, $k_i\in N_k^+(\lambda)$, is indexed by

$$N_k^1(\lambda)=\{k_1,\ldots,k_{\sigma_k}\} \quad \text{for } k=1,\ldots,s.$$

Also, denote

$$N_{s+1}^1(\lambda)=N_{s+1}^+(\lambda), \quad N_k^0(\lambda)=N_k-N_k^1(\lambda) \quad \text{for } k=1,\ldots,s+1, \quad \text{and } N^1(\lambda)=\bigcup_{k=1}^{s+1}N_k^1(\lambda), \quad N^0(\lambda)=\bigcup_{k=1}^{s+1}N_k^0(\lambda).$$

Define $\delta(\lambda)=(\delta_1(\lambda),\ldots,\delta_n(\lambda))$, where

$$\delta_i(\lambda) = \begin{cases} 1 & \text{if } i \in N^1(\lambda) \\ -1 & \text{if } i \in N^0(\lambda). \end{cases}$$

Then for $\lambda = \tilde{\lambda}$,

$$F(\tilde{\lambda}) = \sum_{i \in N^1(\tilde{\lambda})} p_i(\tilde{\lambda}) - p_0(\tilde{\lambda})$$

$$= [-1/2\, \bar{a}\delta(\tilde{\lambda})]\, \tilde{\lambda} + 1/2\, [\bar{x}\, \delta(\tilde{\lambda}) - |N^1(\tilde{\lambda})|]$$

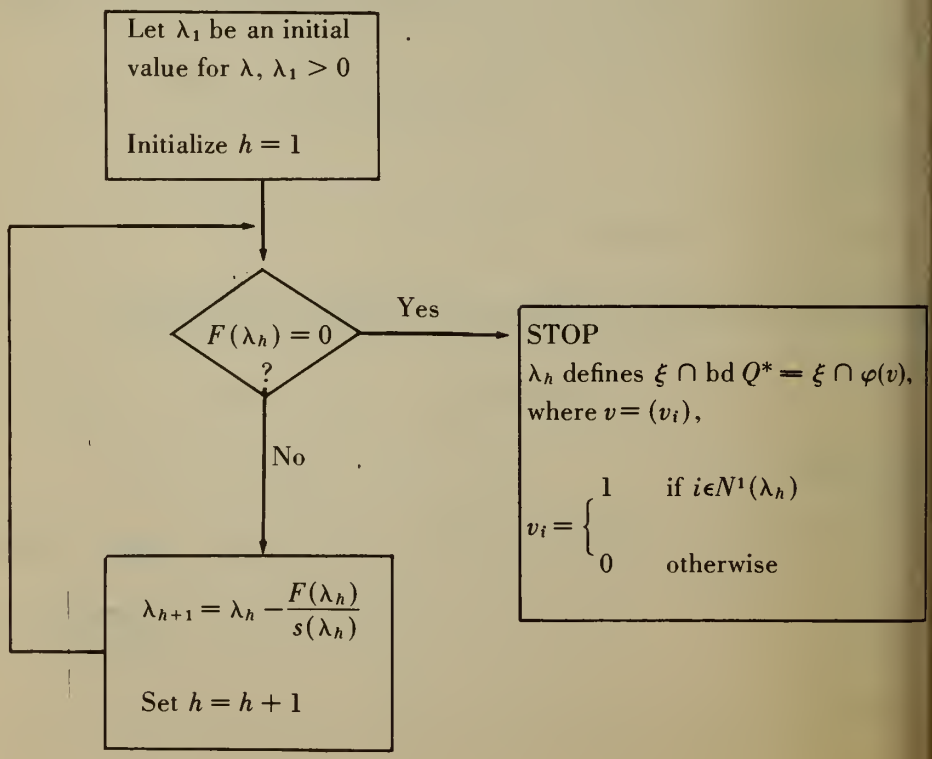
and the slope of $F(\lambda)$ (i.e., $dF/d\lambda$) on some interval containing $\tilde{\lambda}$ is,

$$s(\lambda) = -1/2\bar{a}\delta(\tilde{\lambda}).$$

Notice that the linear segments of $F(\lambda)$ are defined by those values of $\lambda$ for which $N_k^1(\lambda)$ changes. This happens when $\sigma_k = \mu_k < r_k$ and some $p_i(\lambda)$ $(i \in N_k)$ changes sign (i.e., when $N_k^+$ and $N_k^-$ changes); or when $\sigma_k = r_k$ and the set of $r_k$ largest positive coefficients, $p_i(\lambda)$ $(i \in N_k)$, is altered.

The intersection of $\xi$ with bd $Q^*$ can be found by locating the zero of $F(\lambda)$, i.e., solving $F(\lambda) = 0$ for $\lambda \geq 0$. Notice that if $F$ has a zero for $\lambda \geq 0$, it is unique. The method outlined below (Alogrithm 1) can be viewed as a version of Newton's method for solving nonlinear equations, especially adapted for $F(\lambda)$.

ALGORITHM 1:


Let $\lambda_1$ be an initial value for $\lambda$, $\lambda_1 > 0$
Initialize $h = 1$
$F(\lambda_h) = 0$ ?
Yes
STOP
$\lambda_h$ defines $\xi \cap$ bd $Q^* = \xi \cap \varphi(v)$, where $v = (v_i)$,
$v_i = \begin{cases} 1 & \text{if } i \in N^1(\lambda_h) \\ 0 & \text{otherwise} \end{cases}$
No
$\lambda_{h+1} = \lambda_h - \frac{F(\lambda_h)}{s(\lambda_h)}$
Set $h = h + 1$

Since the direction of $\xi$ appears to be more important than its point of origin in determining which facet of $Q^*$ is first intersected (see [3]) and since the facet first intersected by $\bar{\xi} = \{x | x = 1/2\, e - \bar{a}\,\lambda, \lambda \geq 0\}$, the halfline having the same direction as $\xi$, but originating at $1/2\, e$, is trivially easy to find, we choose $\lambda_1$ to be the value defining the intersection of $\bar{\xi}$ with bd $Q^*$.

Let $\bar{N}_k^1(\lambda)$ and $\bar{N}_k^0(\lambda)$ be defined in relation to $\bar{\xi}$ in the same way as $N_k^1(\lambda)$ and $N_k^0(\lambda)$ were in relation to $\xi$. Since the coefficients $p_i(\lambda)$ do not change sign for $\lambda > 0$, the sets $\bar{N}_k^1 = \bar{N}_k^1(\lambda)$ and $\bar{N}_k^0 = \bar{N}_k^0(\lambda)$ never change. Here $\bar{N}_k^1$ is simply the index set for the $\sigma_k$ largest positive coefficients $-\bar{a}_i$, $i \epsilon N_k$.

Let $\bar{N}^1 = \overset{s}{\underset{k=1}{\cup}} \bar{N}_k^1$ and $\bar{N}^0 = \overset{s}{\underset{k=1}{\cup}} \bar{N}_k^0$. Associate $\bar{F}(\lambda)$, $\bar{s} = \bar{s}(\lambda)$ and $\bar{\delta} = \bar{\delta}(\lambda)$ with $\bar{\xi}$. For $\lambda > 0$, $\bar{F}(\lambda)$ is a linear function of $\lambda$ with slope $\bar{s} = -1/2\bar{a}\bar{\delta}$.

If $\bar{s} > 0$ then $\bar{\xi}$ intersects bd $Q^*$ and the facet of $Q^*$ intersected is the one associated with the vertex $v = (v_1, \ldots, v_n)$ of $Q$, where

$$v_i = \begin{cases} 1 & \text{if } i \epsilon \bar{N}^1 \\ 0 & \text{otherwise.} \end{cases}$$

The intersection of $\bar{\xi}$ with this facet (or its extension) occurs for $\lambda = \lambda_1$, where

$$\lambda = \frac{\bar{x}\bar{\delta} - |\bar{N}^1|}{\bar{a}\,\bar{\delta}}. \tag{2.1}$$

Since the set $\bar{N}^1$ is not always unique (there may be ties in the ordering), neither is the facet of $Q^*$ intersected by $\bar{\xi}$, and the associated vertex $v$ of $Q$.

If $\bar{s} \leq 0$ then $\bar{\xi} \cap$ bd $Q^* = \emptyset$. Since $\xi$ and $\bar{\xi}$ originate in the interior of $Q^*$ and have the same direction, they either both intersect bd $Q^*$ or neither does so.

Let $\lambda_1$ be defined by (2.1) and let $\bar{\lambda}$ be the value that we are looking for, i.e., $\xi \cap$ bd $Q^* = \bar{x} - \bar{a}\,\bar{\lambda}$. The following theorem gives an upper bound on the number of steps that Algorithm 1 may require in order to find $\bar{\lambda}$.

THEOREM 2.1: Algorithm 1 finds $\bar{\lambda}$ in at most $r$ iterations, where

$$r = 1/2 \sum_{k=1}^{s} n_k(n_k + 1) + n_{s+1}$$

and $n_k = |N_k|$ for $k = 1, \ldots, s+1$.

PROOF: Let

$$z_k(\lambda) = \max \left\{ \sum_{i \epsilon N_k} p_i(\lambda) x_i \Big| \sum_{i \epsilon N_k} x_i \leq r_k; \quad 0 \leq x_i \leq 1, \quad i \epsilon N_k \right\}, \quad k = 1, \ldots, s$$

and

$$z_{s+1}(\lambda) = \sum_{i \epsilon N_{s+1}} \max \{p_i(\lambda), 0\}.$$

The function $z_k(\lambda)$, $k=1, \ldots, s$, has an inflection point at $\tilde{\lambda}$ only if $p_i(\tilde{\lambda})=p_j(\tilde{\lambda})$ or $p_i(\tilde{\lambda})=0$ for some $i, j \in N_k$. Since there are at most $1/2\ n_k(n_k+1)$ ways for this to occur, $z_k(\lambda)$ has at most $1/2\ n_k(n_k+1)$ inflection points. Since $z_{s+1}(\lambda)$ has at most $n_{s+1}$ inflection points, $z(\lambda)=\sum_{k=1}^{s+1} z_k(\lambda)$ and hence $F(\lambda)$ have at most $r$ inflection points. Since $\lambda_1 \geq \bar{\lambda}$, $F(\lambda_1) \geq 0$, $F(\lambda)$ is a convex piecewise-linear function with at most $r$ inflection points, and $\bar{\lambda}$ is the unique zero for $\lambda \geq 0$, Algorithm 1 finds progressively smaller $\lambda$'s until it converges to $\bar{\lambda}$ in at most $r$ steps. Q.E.D.

The bound given in Theorem 2.1 is a weak one. Typically, only a few iterations are required since the initial value, $\lambda_1$, is usually on one of the first few line segments of $F(\lambda)$, i.e., quite close to $\bar{\lambda}$. Computational experience in the case of the octahedron bears this out [3].

If some halfline $\xi_j$ does not intersect bd $Q^*$, the corresponding coefficient $\bar{\lambda}_j^{-1}$ in the cut (1.1) is set to 0 (which corresponds to $\bar{\lambda}_j=\infty$, i.e., an intersection point "at infinity"). However, according to a recent result [4], outer polar cuts can be strengthened in the direction of such halflines (which do not intersect bd $Q^*$), by allowing $\lambda_j$ to take on a negative value, namely the value for which the halfline $-\xi_j=\{x \mid x=\bar{x}-\bar{a}_j\lambda_j,\ \lambda_j \leq 0\}$ intersects the last supporting hyperplane of $Q^*$. The finding of this value amounts to finding $\hat{\lambda}_j<0$ such that $\hat{F}(\hat{\lambda}_j)=0$, where $\hat{F}(\lambda_j)=\min\left\{\sum_{i\in N} p_i(\lambda_j)x_i \mid x\in Q\right\}-p_0(\lambda_j)$. Since $\hat{F}(\lambda_j)$ is concave and piecewise linear, with $\hat{F}(\lambda_j)<0$ for $\hat{\lambda}_j<\lambda_j\leq 0$, $\hat{F}(\hat{\lambda}_j)=0$ and $\hat{F}(\lambda_j)>0$ for $\lambda_j<\hat{\lambda}_j$, it follows that $\hat{\lambda}_j$ can be found by Algorithm 1.

EXAMPLE: Find the intersection of $\xi=\{x \mid x=\bar{x}-\bar{a}\lambda,\ \lambda \geq 0\}$ with bd $Q^*$, where $n=9$, $\bar{x}=(0.2, 0.5, 0.6, 0, 0, 0.9, 1, 0, 0.9)$, $\bar{a}=(-0.7, 0.5, -0.3, -1, 0, 0.8, 0, 0, 0.3)$, and $Q=\{x \mid x_1+x_2+x_3+x_4 \leq 2,\ x_5+x_6+x_7+x_8 \leq 3\} \cap K$.

1. Find $\lambda_1$:

$$N_1=\{1,2,3,4\}; \quad N_2=\{5,6,7,8\}; \quad N_3=\{9\}; \quad r_1=2; \quad r_2=3.$$

$$\bar{N}_1^+=\{1,3,4\}; \quad \bar{N}_2^+=\bar{N}_3^+=\emptyset; \quad \bar{N}_1^1=\{1,4\}; \quad \bar{N}_2^1=\bar{N}_3^1=\emptyset;$$

$$\delta=(1,-1,-1,1,-1,-1,-1,-1,-1)$$

$$\lambda_1=\frac{-3.7-2}{-1.7-1.3}=1.9.$$

2. Evaluate $F(\lambda_1)$:

$$N_1^+(\lambda_1)=\{1,3,4\}; \quad N_2^+(\lambda_1)=\{7\}; \quad N_3^+(\lambda_1)=\emptyset, N_1^1(\lambda_1)=\{1,4\}; \quad N_2^1(\lambda_1)=\{7\}; \quad N_3^1(\lambda_1)=\emptyset$$

$$\delta(\lambda_1)=(1,-1,-1,1,-1,-1,1,-1,-1)$$

$$F(\lambda_1)=p_1(\lambda_1)+p_4(\lambda_1)+p_7(\lambda_1)-p_0(\lambda_1)=0.5.$$

3. Since $F(\lambda_1)\neq 0$, find $\lambda_2$:

$$s(\lambda_1) = 1.5, \qquad \lambda_2 = \lambda_1 - \frac{F(\lambda_1)}{s(\lambda_1)} = 1.57.$$

4. Evaluate $F(\lambda_2)$:

$N_1^+(\lambda_2)=\{1,3,4\}$; $N_2^+(\lambda_2)=\{7\}$; $N_3^+(\lambda_2)=\emptyset$; $N_1^1(\lambda_2)=\{1,4\}$; $N_2^1(\lambda_2)=\{7\}$; $N_3^1(\lambda_2)=0$.

$F(\lambda_2) = 0$.

Hence $\lambda=1.57$ defines $\xi \cap \text{bd } Q^*$, and $v^1=(1, 0, 0, 1, 0, 0, 1, 0, 0,)$ is the vertex of $Q$ contained in the facet of $Q^*$ intersected by $\xi$.

Note that an alternative stopping rule to "If $F(\lambda_h)=0$ then stop," is "If $N_k^1(\lambda_h)=N_k^1(\lambda_{h-1})$ for $k=1, \ldots, s+1$ then stop." The function $F$ need not be explicitly evaluated for the last iteration if this stopping rule is employed.

## 3. ALGORITHM 2 (FOR FINDING THE EXTENDED FACETS OF $Q^*$ IN THE ORDER IN WHICH THEY ARE INTERSECTED BY $\xi$)

In [3] and [11] algorithms are presented for finding the extended facets of $K^*$ in the order in which they are intersected by $\xi$. Both algorithms can be modified to find the intersection of $\xi$ with the extended facets of $Q^*$. We now state an algorithm based on [11].

Let $\varphi(v^1)$, where $v^1 \in \text{vert } Q$, be the facet of bd $Q^*$ intersected by $\xi$, and let $\lambda^1$ be the value of $\lambda$ defining the intersection point. We wish to find the values $\lambda^2, \lambda^3, \ldots, \lambda^k$ of $\lambda$, defining the intersection of $\xi$ with the 2nd, 3rd, . . ., $k$th extended facet of $Q^*$.

The following partition of vert $Q-\{v^1\}$ induces a partition of the extended facets of $Q^*$ other than $\varphi(v^1)$:

$$\text{vert } Q-\{v^1\} = \bigcup_{h=1}^{n} V^h,$$

where $V^h = Q \cap \text{vert } F_h$, $h=1, \ldots, n$, and where

$$F_h=\{x \in K \mid x_i=v_i^1, \quad i=1, \ldots, h-1, \quad x_h=1-v_h^1\}. \tag{3.1}$$

Then each $F_h$, $h=1, \ldots, n$, is an $(n-h)$-dimensional face of $K$, the faces $F_1, \ldots, F_n$ are disjoint, and $F_h \cap Q$ is a truncated $(n-h)$-dimensional cube contained in aff $F_h$, the affine hull of $F_h$. Let $(F_h \cap Q)^*$ be the outer polar of $F_h \cap Q$, defined in terms of the subspace containing $F_h$. Let $\xi^h$ be the projection of $\xi$ onto aff $F_h$. Then the facet of $(F_h \cap Q)^*$ intersected by $\xi^h$ is $f^h \cap$ aff $F_h$, where $f^h$ is the first extended facet of $Q^*$ intersected by $\xi$ such that $\varphi^{-1}(f^h)=\bar{v}^h \in V^h$.

Each face $F \in \{F_1, \ldots, F_n\}$ can be defined via the two index sets $N_F^1=\{i \in N \mid x_i=1\}$, $N_F^0=\{i \in N \mid x_i=0\}$, given by the condition $x_i=v_i^1$, $i=1, \ldots, h-1$, $x_h=1-v_h^1$ (where $F=F_h$). Denoting $N_F=N_F^1 \cup N_F^0$ (the set of "free" variables), the intersection point $\xi^h \cap \text{bd } (F \cap Q)^*$, is found by solving for $\lambda$ the following problem, which is a slight modification of (PLP) of section 1, and can therefore be solved by Algorithm 1:

$$\max \sum_{i \in N_F} p_i(\lambda) x_i = p_0(\lambda) - \sum_{i \in N_F^1} p_i(\lambda)$$

$$\text{s.t.} \quad \sum_{i \in N_F \cap N_k} x_i \leq r_k - |N_F^1 \cap N_k| \qquad \text{for } k = 1, \ldots, s$$

$$0 \leq x_i \leq 1 \qquad \text{for } i \in N_F.$$

Let $\bar{\lambda}^h$ define the intersection of $\xi^h$ with the facet aff $F_h \cap \varphi(\bar{v}^h)$ of $(F_h \cap Q)^*$. Then $\lambda^2 = \min_{h=1,\ldots,u} \{\bar{\lambda}^h\}$ defines the second extended facet of $Q^*$ intersected by $\xi$. More generally, if the first $t$ extended facets of $Q^*$ intersected by $\xi$ have been found, and $\lambda^t$ defines the $t$th one, the $(t+1)$st extended facet can be found by the procedure formalized in the next theorem and the subsequent algorithm.

THEOREM 3.1: Let $\lambda^1, \ldots, \lambda^t$ define the first $t$ extended facets of $Q^*$ intersected by $\xi$ and let $v^1, \ldots, v^t$ be the associated vertices of $Q$. Let $\{V^h\}_{h=1,\ldots,u}$ be a partition of vert $Q - \{v^1, \ldots, v^t\}$ and let $\{F_h\}_{h=1,\ldots,u}$ be a set of disjoint faces of $K$ such that $V^h \subseteq F^h$. Further, let $\xi^h$ be the projection of $\xi$ on aff $F_h$. If $\bar{\lambda}^h$ defines the intersection of $\xi^h$ with bd $(F_h \cap Q)^*$, then $\lambda^{t+1} = \min_{h=1,\ldots,u} \{\lambda^h\}$ defines the $(t+1)$st extended facet of $Q^*$ intersected by $\xi$.

PROOF: Since $\bar{\lambda}^h$ defines the extended facet of $Q^*$ in $\{\varphi(\bar{v})\}_{v \in V^h}$ which is first intersected by $\xi$, it follows that $\lambda^{t+1} = \min_{h=1,\ldots,u} \{\bar{\lambda}^h\}$ defines the extended facet of $Q^*$ in $\{\varphi(\bar{v}^h)\}_{h=1,\ldots,u}$ first intersected by $\xi$. Since $\{V^h\}_{h=1,\ldots,u}$ forms a partition of vert $Q - \{v^1, \ldots, v^t\}$, the $(t+1)$st facet intersected is defined by $\lambda^{t+1}$. Q.E.D.

Algorithm 2 obtains the intersection of $\xi$ with successive facets of $Q^*$ in the order of intersection.

ALGORITHM 2:

0. Set $F_1 = K$, $\Lambda = \emptyset$, $k = 1$, $u = 0$, $\eta = 1$. Go to 2.

1. Let $N_{F_t} = N - N_{F_t}^1 \cup N_{F_t}^0 = \{i_1, \ldots, i_\eta\}$.
   Partition vert $(F_t \cap Q) - \{\bar{v}^t\}$ as follows:
   vert $(F_t \cap Q) - \{\bar{v}^t\} = \bigcup_{j=1}^{\eta} V^{u+j}$,
   where $V^{u+j} = Q \cap$ vert $F_{u+j}$
   and $F_{u+j} = \{x \in K | x_i = 1\ i \in N_{F_t}^1;\ x_i = 0, i \in N_{F_t}^0;\ x_i = \bar{v}_i^t\ i = 1, \ldots, i_{j-1};\ x_{i_j} = 1 - \bar{v}_{i_j}^t\}$
   for $j = 1, \ldots, \eta$.
   Go to 2.

2. Find $\xi^h \cap$ bd $(F_h \cap Q)^*$ via Algorithm 1 for $h = u+1, \ldots, u+\eta$.
   For each $h$ let $\varphi(\bar{v}^h)$ be the facet intersected and let $\bar{\lambda}^h$ define the intersection.
   Set $\Lambda \leftarrow \Lambda \cap \{\bar{\lambda}^{u+1}, \ldots, \bar{\lambda}^{u+\eta}\}$
   If $\xi^h \cap$ bd $(F_h \cap Q)^* = \emptyset$, vertices $V^h$ need not be considered any more, i.e., $\xi \cap \varphi(\bar{v}) = \emptyset$ *for* $\bar{v} \in V^h$.
   Go to 3.

3. Determine $\bar{\lambda} = \bar{\lambda}^t = \min \{\bar{\lambda}^h | \bar{\lambda}^h \in \Lambda\}$
   If $\Lambda = \emptyset$ then $\xi$ does not intersect any more extended facets of $Q^*$. Otherwise, $\bar{\lambda}^t$ and $\varphi(\bar{v}^t)$ determine the intersection of $\xi$ with the $k$th extended facet of $Q^*$, i.e., $\lambda^k = \bar{\lambda}^t$ and $v^k = v^t$. (The algorithm can be stopped here at any time.)

$\Lambda \leftarrow \Lambda - \{\bar{\lambda}^t\}$
$u \leftarrow u + \eta$
$k \leftarrow k+1$
Go to 1.

We note that, in accordance with the discussion at the end of section 1, if $\Lambda = \emptyset$ a valid coefficient $\hat{\lambda}^{-1} < 0$ for the cut (1.1) can be obtained by finding the value $\hat{\lambda} < 0$ for which the halfline $-\xi$ intersects the last supporting hyperplane of $Q^*$. If one is interested in further strengthening the cut, a sequence of coefficients $0 > \hat{\lambda}^1 > \hat{\lambda}^2 \ldots$ can be calculated by finding the intersection of $-\xi$ with the last, next to last, etc. supporting hyperplanes of $Q^*$. Algorithm 2 can be easily modified to do this.

EXAMPLE (continued): Algorithm 2 is used to find the values of $\lambda$ defining the intersection of the halfline $\xi$ with the next five extended facets of $Q^*$ in the example of section 2.

*Iteration 1:*

1. Partition: vert $Q - \{v^1\} = \bigcup_{j=1}^{9} V^j$, where $V^j = Q \cap$ vert $F_j$ and

$$F_1 = \{x \in K | x_1 = 0\}$$

$$F_2 = \{x \in K | x_1 = 1, x_2 = 1\}$$

$$\cdot$$
$$\cdot$$

$$F_j = \{x \in K | x_i = v_i^1, i = 1, \ldots, j-1, x_j = 1 - v_j^1\}$$

$$\cdot$$
$$\cdot$$

$$F_9 = \{x \in K | x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 1, x_5 = 0, x_6 = 0, x_7 = 1, x_8 = 0, x_9 = 1\}.$$

2. Algorithm 1 is used to find $\xi^j \cap \text{bd}\,(F_j \cap Q)^*$ $(j = 1, \ldots, 9)$.
Let $\bar{\lambda}^j$ and $\bar{v}^j$ define this intersection. Then

$\bar{\lambda}^1 = 1.77$ $\quad \bar{v}^1 = (0, 0, 1, 1, 0, 0, 0, 0, 0)$
$\bar{\lambda}^2 = \infty$
$\bar{\lambda}^3 = 2.19$ $\quad \bar{v}^3 = (1, 0, 1, 0, 0, 0, 0, 0, 0)$
$\bar{\lambda}^4 = 3.70$ $\quad \bar{v}^4 = (1, 0, 0, 0, 0, 0, 0, 0, 0)$
$\bar{\lambda}^5 = 1.90$ $\quad \bar{v}^5 = (1, 0, 0, 1, 1, 0, 0, 0, 0)$
$\bar{\lambda}^6 = 2.77$ $\quad \bar{v}^6 = (1, 0, 0, 1, 0, 1, 0, 0, 0)$
$\bar{\lambda}^7 = 1.90$ $\quad \bar{v}^7 = (1, 0, 0, 1, 0, 0, 1, 0, 0)$
$\bar{\lambda}^8 = 1.63$ $\quad \bar{v}^8 = (1, 0, 0, 1, 0, 0, 0, 1, 0)$
$\bar{\lambda}^9 = 1.90$ $\quad \bar{v}^9 = (1, 0, 0, 1, 0, 0, 0, 0, 1)$

3. The intersection with the second extended facet is defined by

$\lambda^2 = \min\{\bar{\lambda}^1, \ldots, \bar{\lambda}^9\} = \bar{\lambda}^8 = 1.63$ and $\bar{v}^8 = (1, 0, 0, 1, 0, 0, 0, 1, 0)$.

*Iteration 2:*

1. Partition: vert $(F_8 \cap Q) - \{\bar{v}^8\} = V^{10}$, where $V^{10} = Q \cap$ vert $F_{10}$, and

$$F_{10} = f_8 \cap \{x \in K | x_9 = 1\}.$$

2. Algorithm 1 is used to find $\xi^{10} \cap \text{bd}\,(F_{10} \cap Q)^*$

$$\bar{\lambda}^{10} = 2.04\,, \bar{v}^{10} = (1, 0, 0, 1, 0, 0, 0, 1, 1).$$

3. The third extended facet intersected by $\xi$ is the one defined by

$$\lambda^3 = \min\{\bar{\lambda}^1, \ldots, \bar{\lambda}^7, \bar{\lambda}^9, \bar{\lambda}^{10}\} = \bar{\lambda}^1 = 1.77 \text{ and } \bar{v}^1 = (0, 0, 1, 1, 0, 0, 0, 0, 0).$$

*Iteration 3:*

1. Partition: vert $(F_1 \cap Q) - \{\bar{v}^1\} = \bigcup_{j=1}^{8} V^{10+j}$, where $V^{10+j} = Q \cap \text{vert } F_{10+j}$, and

$$F_{10+j} = \{x \in K \mid x_1 = 0,\ x_1 = \bar{v}_i^1,\ i = 2, \ldots, j;\ x_i = 1 - \bar{v}_j^1,\ i = j+1\},\ j = 1, \ldots, 8.$$

2. Algorithm 1 is used to find $\xi^j \cap \text{bd}\,(F_j \cap Q)^*$ for $j = 11, \ldots, 18$.

$\bar{\lambda}^{11} = 6.83$ $\bar{v}^{11} = (0, 1, 0, 1, 0, 0, 0, 0, 0)$
$\bar{\lambda}^{12} = 3.19$ $\bar{v}^{12} = (0, 0, 0, 1, 0, 0, 0, 0, 0)$
$\bar{\lambda}^{13} = 14.5$ $\bar{v}^{13} = (0, 0, 1, 0, 0, 0, 0, 0, 0)$
$\bar{\lambda}^{14} = 2.23$ $\bar{v}^{14} = (0, 0, 0, 1, 1, 1, 0, 0, 0)$
$\bar{\lambda}^{15} = 5.17$ $\bar{v}^{15} = (0, 0, 1, 1, 0, 1, 0, 0, 0)$
$\bar{\lambda}^{16} = 2.23$ $\bar{v}^{16} = (0, 0, 1, 1, 0, 0, 1, 0, 0)$
$\bar{\lambda}^{17} = 1.94$ $\bar{v}^{17} = (0, 0, 1, 1, 0, 0, 0, 1, 0)$
$\bar{\lambda}^{18} = 2.23$ $\bar{v}^{18} = (0, 0, 1, 1, 0, 0, 0, 0, 1)$

3. The intersection with the fourth, fifth, and sixth extended facets occur when $\lambda^4 = \lambda^5 = \lambda^6 = \min\{\bar{\lambda}^2, \ldots, \bar{\lambda}^7, \bar{\lambda}^9, \ldots, \bar{\lambda}^{18}\} = \bar{\lambda}^5 = \bar{\lambda}^7 = \bar{\lambda}^9 = 1.90$. The associated vertices are $\bar{v}^5, \bar{v}^7$ and $\bar{v}^9$.

## 4. ALGORITHM 3 (FOR GENERATING A SUITABLE TRUNCATED CUBE)

In Sections 2 and 3 we discussed how one can determine the intersection of a halfline with the boundary and extended facets of the outer polar, $Q^*$, of a truncated cube, $Q$. Based on these points of intersection, valid intersection cuts can be generated for (P). We now introduce truncated cube intersection cuts derived by using for each halfline $\xi_j$, $j \in J$, a different truncated cube $Q_j$, specially tailored for $\xi_j$. In addition, we show how valid intersection cuts can be obtained by intersecting each $\xi_j$ with the extended facets of several truncated cubes.

For any halfline $\xi_j$, the depth of the intersection $\xi_j \cap \text{bd}\,Q^*$ depends upon the choice of $Q$. But a truncated cube $Q$ best suited for a given halfline may be very ill-suited for another one. The following theorem shows that it is possible to generate valid intersection cuts from the intersection of each halfline $\xi_j$ with the boundary (and extended facets) of a different set $Q_j^*$, where $Q_j$ is a truncated cube tailored to halfline $\xi_j$.

THEOREM 4.1: Let $\{Q_j\}_{j \in J}$ be a set of truncated cubes, each of which contains all integer points of $X$. Then the inequality (1.1), where each $\bar{\lambda}_j$ defines the point $\alpha_j = \xi_j \cap \text{bd}\,Q_j^*$, is a valid (intersection) cut for (P).

PROOF: Let $\hat{X} = \text{conv}\{x \in X | x \text{ integer}\}$. Since $X \subseteq (\bigcap_{j \in J} Q_j)$, from the basic properties of polar sets we have $(\bigcup_{j \in J} Q_j^*) \subseteq \hat{X}^*$. Since $\hat{X}^*$ was shown in [2] to give rise to valid intersection cuts, the theorem follows. Q.E.D.

Since any truncated cube $Q$ is the convex hull of a subset of the vertices of $K$, i.e., vert $Q \subset$ vert $K$, from property (vii) of outer polars listed in section 1, it follows that the facets of $Q^*$ are a subset of the facets of $K^*$. Therefore, in choosing a convenient truncated cube $Q_j$ for a given halfline $\xi_j$, one may want to make sure that $\xi_j$ intersects as few of the extended facets of $K^*$ which are also extended facets of $Q_j^*$, as possible. In particular, a reasonable requirement on $Q_j^*$ is that the facet of $K^*$ intersected by $\bar{\xi}_j$ (the halfline originating at 1/2 $e$ with same direction as $\xi_j$) should not be a facet of $Q_j^*$. That is, among the constraints that define $Q_j$, there should be some that cut off the vertex $\tilde{v}$ of $K$, associated with this facet of $K^*$. If no canonical cut can be found which cuts off $\tilde{v}$, then $\tilde{v} \in X$, i.e., $\tilde{v}$ is a feasible integer point.

If deeper intersection cuts are to be derived via intersecting several extended facets and checking the corresponding vertices of $K$ for feasibility, it may even be advantageous to associate more than one truncated cube with each halfline. Recall that at any point in Algorithm 2, the set of facets not yet intersected is partitioned. Let $V^h$ be one of the members in the corresponding vertex partition and let $F_h$ be the associated face of $K$. Then the first facet of $Q^*$ in $\varphi(V^h)$ intersected by $\xi$ is obtained by determining $\xi^h \cap \text{bd}\ (Q \cap F_h)^*$, where $\xi^h$ is, as before, the projection of $\xi$ on aff $F_h$. Valid intersection cuts can be derived if legitimate truncated cubes other than $Q$ are used for the partition. Hence truncated cubes, $Q_{F_h}$, may be chosen which may yield a deeper intersection, $\xi^h \cap \text{bd}\ (Q_{F_h} \cap F_h)^*$, than does $Q$. A reasonable choice for $Q_{F_h}$ is a truncated cube which cuts off the vertex of $F_h$ associated with the facet of $F_h^*$ intersected by $\bar{\xi}^h$. Note that the extended facets intersected by $\xi$ are extended facets of different specially selected truncated cubes.

To see that such an intersection cut is valid, one has only to observe that $\hat{\hat{X}} \subseteq (\bigcap_h Q_{F_h})$, where the intersection is to be taken over all truncated cubes $Q_{F_h}$ used in the procedure, and where $\hat{\hat{X}} = \text{conv}\{x \in X | x \text{ integer}, cx \geq c\hat{x}\}$, $\hat{x}$ being the best feasible vertex found so far. Thus the above Theorem for the sets $Q_j$, $j \in J$, also applies to the sets $Q_{F_h}$.

As an illustration of this second approach, consider once again the example presented in Sections 2 and 3. Assume that it is possible to obtain the truncated cube $Q_{F_1} = \{x | x_1 = 0,\ x_3 + x_4 \leq 1\} \cap K$ for partition 1. Then $\bar{\lambda}^1 = 2.81$ and $\bar{v}^1 = (0, 0, 0, 1, 0, 0, 1, 0, 0)$, showing that $Q_{F_1}$ produces a deeper intersection than $Q$.

We now turn to the problem of generating suitable truncated cubes.

Many practical applications of integer programming involve multiple choice constraints, i.e., disjoint constraints of the form $\sum_M x_i = k$ or $\sum_M x_i \leq k$, where $M \subseteq N$ and $1 \leq k \leq |M| - 1$ is an integer. In such cases, a truncated cube is readily available as the intersection of $K$ with the hyperplanes or half-spaces defined by the multiple choice constraints. Generally, however, the truncated cube is not readily available. To create it, we generate canonical cuts (see [5]) derived from the problem constraints. Any inequality in 0–1 variables can be brought to the "rectified" form

$$\sum_{j \in N} a_j x_j \leq a_0, \tag{4.1}$$

where $a_0 > 0$, $a_j \geq 0$, $j \in N$, by complementing some variables if necessary. Let $\mathcal{S}$ be the family of all subsets $S \subseteq N$ such that

$$\sum_{j \in S} a_j > a_0 \text{ but } \sum_{j \in S - \{i\}} a_j \leq a_0, \qquad \forall i \in S,$$

and for each $S \in \mathcal{S}$, define

$$S' = \{j \in N - S \mid a_j \geq a_i, \qquad \forall i \in S\}.$$

It can easily be seen that for any $S \in \mathcal{S}$, the inequality

$$\sum_{j \in S \cup S'} x_j \leq |S| - 1 \tag{4.2}$$

is satisfied by all $x \in \{0, 1\}^n$ satisfying (4.1). Furthermore, it was shown in [5] that the family of inequalities (4.2), termed canonical, having one member for each $S \in \mathcal{S}$, is equivalent to (has the same 0–1 solutions as) the inequality (4.1).

Inequalities of the form (4.2) are easy to generate from the problem constraints, and if they are disjoint, they define a truncated cube. However, in general, the number of such canonical inequalities implied by the problem constraints is very large, and the problem of finding a convenient subset, i.e., one whose members are disjoint and cut off a specified vertex as "strongly" as possible, requires special attention. Next we discuss an algorithm for doing this.

Let $M$ be the index set for the problem constraints $a^h x \leq b_h$ (other than the facets of $K$), including $-cx \leq -z^* - 1$, where $z^*$ is the current best value obtained for $cx$.

Let $N_1, \ldots, N_s$ be the (disjoint) index sets of the variables having nonzero coefficients in the $s$ canonical cuts generated so far, and let $N_{s+1} = N - \bigcup_{i=1}^{s} N_i$.

At each stage of the algorithm to be described, a problem constraint is selected, which cuts off as "strongly" as possible, or comes as close as possible to cutting off, the vertex of the current truncated cube $Q$ contained in the facet of $Q^*$ intersected by $\xi$. This constraint is then rectified, and from among the canonical inequalities implied by it, one with the smallest right hand side is selected. Next, in order to insure the disjointness of the set of canonical inequalities that is generated, all problem constraints are "cleared" of the variables having a positive coefficient in the inequality just produced. This is done so as to relax the problem constraints as little as possible, by making full use of the information in the latest canonical constraint. Then a new problem constraint is selected, etc., until either $N_1 \cup \ldots \cup N_s = N$, or no more canonical cuts can be generated.

0. *Initialize.* Set $Q = K$, $s = 0$, $N_{s+1} = N$. Go to 1.

1. *Select a problem constraint*

Let $v$ be the vertex of $Q$ contained in $\varphi(v)$, the facet of $Q^*$ intersected by $\bar{\xi} = \left\{ x \mid x = \frac{1}{2} e - \bar{a}\lambda, \lambda \geq 0 \right\}$, and let $\hat{v}$ be the projection of $v$ on the subspace indexed by $N_{s+1}$. If $v \in X \cap \{x \mid cx \geq z^* + 1\}$, a new (better) feasible 0–1 point has been found. Store $v$ and update $z^*$ (set $z^* = cv$).

Let $\hat{a}^h\hat{x} \leqslant \hat{b}_h$, $h \in M$, be the problem constraints at the current stage. That is, initially $\hat{a}^h = a^h$, $\hat{x} = x$, $\hat{b}_h = b_h$; later, when $N_{s+1} \neq N$, $\hat{a}^h$ and $\hat{x}$ are the projections of $a^h$ and $x$ on the subspace indexed by $N_{s+1}$, and $\hat{b}_h$ is the right hand side of the problem constraint "cleared" of the variables $x_j, j \in N - N_{s+1}$.

Choose an inequality $\hat{a}^k\hat{x} \leqslant \hat{b}_k$ such that

$$\frac{\hat{a}^k\hat{v} - \hat{b}_k}{|\hat{b}_k|} = \max_{h \in M}\left\{\frac{\hat{a}^h\hat{v} - \hat{b}_h}{|\hat{b}_h|}\right\}. \tag{4.3}$$

Go to 2.

2. *Generate a canonical inequality*

"Rectify" the inequality selected under step 1, i.e., complement all variables having negative coefficient, and let $\tilde{a}^k\tilde{x} \leqslant \tilde{b}_k$ be the rectified inequality. Let $\tilde{v}$ be obtained from $\hat{v}$ by complementing the same components as in $\hat{x}$.

Let $N' = N_{s+1}$ if $\tilde{a}^k\tilde{v} \leqslant \tilde{b}_k$; otherwise let $N'$ be the subset of those $j \in N_{s+1}$, such that $\tilde{v}_j = 1$. In both cases, order $N' = \{i_1, \ldots, i_{|N'|}\}$ so that if $h < j$ then $a_{ih} \geqslant a_{ij}$. Let $i_p$ be the first element of $N'$, if it exists, such that

$$\sum_{r=1}^{p} \tilde{a}^k_{i_r} > \tilde{b}_k, \qquad \{i_1, \ldots, i_p\} \subseteq N'$$

and let $i_q$ be the last element of $N'$, such that

$$\sum_{r=q-p+1}^{q} \tilde{a}^k_{i_r} > \tilde{b}_k, \qquad \{q-p+1, \ldots, i_q\} \subseteq N'.$$

Finally, let $\{j_1, \ldots, j_t\}$ be the set of those $i \in N_{s+1} - N'$ such that $\tilde{a}^k_i \geqslant \tilde{a}^k_{i_{q-p+1}}$ (this set is empty when $\tilde{a}^k\tilde{v} \leqslant \tilde{b}_k$).

Then generate the canonical inequality

$$\sum_{i \in T} \tilde{x}_i \leqslant p - 1, \qquad \text{where } T = \{i_1, \ldots, i_q\} \cup \{j_1, \ldots, j_t\},$$

and go to 3.

If $i_p$ does not exist, replace $\tilde{a}^k\tilde{x} \leqslant \tilde{b}_k$ by another inequality chosen by the same criterion (4.3). and apply to it the procedure described in this step. If all constraints have been exhausted and no canonical inequality can be generated, terminate: $Q_j$ is the subset of $K$ defined by the canonical inequalities generated by Algorithm 3.

3. *"Clear" the constraints of the variables indexed by* T

For each problem constraint $\tilde{a}^k\tilde{x} \leqslant \tilde{b}_k$ (expressed in the same variables as (4.4)), define

$$T_k^- = \{i \in T \mid \tilde{a}^k_i < 0\},$$

and let $T_k^*$ be the index set for the $t$ most negative (i.e., smallest) coefficients of $T_k^-$, where $t = \min\{p-1, |T_k^-|\}$. Since (4.4) implies that at most $p-1$ of the variables indexed by $T$ can take on the value 1, any 0–1 vector in the subspace indexed by $N_{s+1}-T$ which has a completion in $N_{s+1}$ satisfying $\tilde{a}^k\tilde{x} \leq \tilde{b}_k$ and (4.4), must satisfy

$$\sum_{i \in N_{s+1}-T} \tilde{a}_i^k \tilde{x}_i \leq \tilde{b}_k - \sum_{i \in T_k^*} \tilde{a}_i^k. \tag{4.5}$$

Hence (4.5) is the new inequality to replace $\tilde{a}^k\tilde{x} \leq \tilde{b}_k$.

Apply step 3 to each problem constraint, set

$$N_{s+2} \leftarrow N_{s+1}-T,$$

$$N_{s+1} \leftarrow T,$$

$$s \leftarrow s+1,$$

and go to 1.

Let $\tilde{v}$ be defined as in Algorithm 3.

THEOREM 4.2: If there exists a constraint

$$\tilde{a}^k\tilde{x} \leq \tilde{b}_k, \tag{4.6}$$

which cuts off $\tilde{v}$, then the canonical inequality (4.4) generated from (4.6) by step 2 of Algorithm 3 has the following properties:

(i) it cuts off $\tilde{v}$;

(ii) it has the smallest right-hand side $(p-1)$ among all canonical inequalities implied by (4.6), which satisfy (i);

(iii) it has the largest number of positive coefficients among all canonical inequalities implied by (4.6), which satisfy (i) and (ii).

PROOF:

(i) From $\tilde{a}^k\tilde{v} > \tilde{b}_k$ and the definitions of $N'$, $p$ and $T$ in step 2 of the Algorithm, it follows that $\{i_1, \ldots, i_p\} \subseteq T$ and

$$\sum_{i \in T} \tilde{v}_i > p-1$$

i.e., (4.4) cuts off $\tilde{v}$.

(ii) By definition, $p$ is the cardinality of the smallest subset $s \subset N'$ such that

$$\sum_{j \in S} \tilde{a}_j^k > \tilde{b}_k; \tag{4.7}$$

hence $p-1$ is the smallest possible right-hand side for a canonical inequality implied by (4.6) and defined on the index set $N'$ (i.e., satisfying (i)).

(iii) Also by definition, $q$ is the largest element of $N'$ such that (4.7) holds for $S=\{i_{q-p+1}, \ldots, i_q\}$. Therefore $T$ is the largest possible subset of $N'$ for which (4.4) is a valid inequality (that cuts off $\tilde{v}$ and has right-hand side $p-1$). Q.E.D.

The following numerical example illustrates how canonical inequalities which cut off vertex $\tilde{v}$ are generated in step 2 and how constraints are "cleared" in step 3 of Algorithm 3.

EXAMPLE: Consider the constraint

$$-10\hat{x}_1-10\hat{x}_2-8\hat{x}_3-8\hat{x}_4+6\hat{x}_5+5\hat{x}_6+4\hat{x}_7+2\hat{x}_8+\hat{x}_9 \leq -11 \tag{4.6}$$

and the vertex $\hat{v}=(1, 0, 0, 0, 0, 0, 1, 1, 1)$ of $K$. To generate a first canonical inequality which cuts off $\hat{v}$, rectify (4.6) and complement $\hat{v}$ accordingly:

$$10\tilde{x}_1+10\tilde{x}_2+8\tilde{x}_3+8\tilde{x}_4+6\tilde{x}_5+5\tilde{x}_6+4\tilde{x}_7+2\tilde{x}_8+\tilde{x}_9 \leq 25$$

$$\tilde{v}=(0, 1, 1, 1, 0, 0, 1, 1, 1)$$

$$N'=\{2, 3, 4, 7, 8, 9\}.$$

Hence, $p=3$, $q=3$, $i_3=4$, $T=\{2, 3, 4\} \cup \{1\}$, and the desired canonical constraint expressed in terms of the complemented variables is

$$\tilde{x}_1+\tilde{x}_2+\tilde{x}_3+\tilde{x}_4 \leq 2.$$

Another problem constraint, say

$$5\hat{x}_1+6\hat{x}_2+7\hat{x}_3+2\hat{x}_4+12\hat{x}_5+12\hat{x}_6+4\hat{x}_7+14\hat{x}_8-5\hat{x}_9 \leq 20, \tag{4.7}$$

or

$$-5\tilde{x}_1-6\tilde{x}_2-7\tilde{x}_3-2\tilde{x}_4+12\tilde{x}_5+12\tilde{x}_6+4\tilde{x}_7+14\tilde{x}_8-5\tilde{x}_9 \leq 0$$

is "cleared" of the variables indexed by $T=\{1, 2, 3, 4\}$ as follows:

$$T_k^-=\{1, 2, 3, 4\},\ T_k^*=\{2, 3\}.$$

Hence, the new constraint is:

$$12\hat{x}_5+12\hat{x}_6+4\hat{x}_7+14\hat{x}_8-5\hat{x}_9 \leq 13.$$

## 5. RELATION TO GLOVER'S CUT FOR 0–1 PROGRAMS WITH A MULTIPLE CHOICE CONSTRAINT

Glover has shown [9] that in the case of a problem whose constraints include a "multiple choice" equation

$$\sum_{i\in M} x_i = k, \tag{5.1}$$

where $M \subseteq N$, $|M| = m$ and $1 \leq k \leq m-1$, a valid intersection cut (convexity cut) can be obtained from the "deformed octahedron"

$$S = \left\{ x \in R^n \mid \sum_{i\in M} |(1+v)x_i - 1/2| \leq m/2 + vk \right\}, \tag{5.2}$$

where $v$ is a scalar satisfying $v > -1/2$.

The next Theorem shows that the outer polar of the truncated cube defined by the "multiple choice" constraint (5.1) gives rise to intersection cuts uniformly stronger than $S$.

Let $x = (x^m, x^{n-m}) \in R^m \times R^{n-m}$, where $x^m$ and $x^{n-m}$ are indexed by $M$ *and* $N-M$, respectively. Consider the truncated $m$-dimensional cube

$$Q_m = \left\{ x^m \in R^m \mid 0 \leq x^m \leq e^m, \sum_{i\in M} x_i = k \right\},$$

its outer polar

$$Q_m^* = \{ y^m \in R^m \mid (x^m - 1/2\, e^m)\,(y^m - 1/2\, e^m) \leq m/4, \quad \forall x^m \in Q_m \},$$

and the $n$-dimensional extension of $Q_m^*$,

$$\begin{aligned} Q_{m(n)}^* &= Q_m^* + R^{n-m} \\ &= \{ y \in R^n \mid y = (y^m, y^{n-m}), \quad (x^m - 1/2\, e^m)\,(y^m - 1/2\, e^m) \leq m/4, \forall x^m \in Q_m \}. \end{aligned}$$

THEOREM 5.1: The intersection cut from the set $Q_{m(n)}^*$ is uniformly stronger than the one from $S$.

PROOF:

(i) We first show that

$$S = \left\{ x \in R^n \mid \sum_{i\in P} x_i - \sum_{i\in Q} x_i \leq \frac{kv+p}{v+1}, \quad \forall (P,Q) \in \mathcal{P} \right\}, \tag{5.3}$$

where $\mathcal{P}$ is the set of all possible partitions of $M$ into two nonempty subsets $P$, $Q$, and $p = |P|$.

Indeed, denoting

$$P = \{ i \in M \mid (1+v)x_i \geq 1/2 \}, \qquad Q = \{ i \in M \mid (1+v)x_i < 1/2 \},$$

and $p=|P|$, $q=|Q|$, the inequality defining Glover's convex set $S$ becomes

$$\sum_{i\in M}|(1+v)x_i-1/2|=\sum_{i\in P}(1+v)x_i-p/2-\sum_{i\in Q}(1+v)x_i+q/2$$

$$=(1+v)\left(\sum_{i\in P}x_i-\sum_{i\in Q}x_i\right)-p+m/2\leq m/2+vk,$$

which, after dividing through with $1+v$ (always positive in view of $v>-1/2$), yields the inequality in (5.3). Since $S$ is the set of all $x\in R^n$ for which the inequality in 5.2 holds, the inequality of (5.3) has to hold for all possible partitions $(P, Q)$ of $M$, i.e., for all $(P, Q)\in\mathscr{P}$.

(ii) Next we show that

$$Q^*_{m(n)}=\left\{x\in R^n|\ \sum_{i\in P}x_i-\sum_{i\in Q}x_i\leq k,\qquad \forall\ (P, Q)\in\mathscr{P}(k)\right\},\tag{5.4}$$

where $\mathscr{P}(k)$ is the set of all partitions $(P, Q)$ of $M$ such that $|P|=p=k$.

Indeed, $K^*_m$, the $m$-dimensional octahedron, is the intersection of all halfspaces defined by

$$\sum_{i\in P}x_i-\sum_{i\in Q}x_i\ \leq|P|,\quad \forall\ (P, Q)\in\mathscr{P}\tag{5.5}$$

and $Q^*_m$ is the intersection of those among the halfspaces defining $K^*_m$, whose defining hyperplanes contain the vertices of $K_m$ satisfying $\sum_{i\in M}x_k=k$; in other words, those halfspaces of the form (5.5), for which $|P|=k$.

Hence $Q^*_{m(n)}$ is of the form (5.4).

(iii) From (5.3) and (5.4) it follows that $Q^*_{m(n)}$ is the intersection of a proper subset of the halfspaces defining $S$, namely those corresponding to partitions $(P, Q)\in\mathscr{P}$ such that $|P|=p=k$. Indeed, for $p=k$ the righthand side of the inequality in (5.3) becomes $k$, for any finite value of $v$.

Hence $S\subset Q^*_{m(n)}$, where the inclusion is proper, since all those facets of $S$ corresponding to partitions $(P, Q)\in\mathscr{P}$, such that $|P|=p\neq k$ cut off nonempty subsets of $Q^*_{m(n)}$. Q.E.D

As an illustration that the halflines $\xi_j$ may intersect facets of $S$ that are not facets of $Q^*_{m(n)}$, consider the following example:

$$\max x_1$$

$$\text{s.t. } x_1-x_2\leq 0$$

$$x_2-x_3\leq 0$$

$$x_1+x_2+x_3=1$$

$$x_j=0 \text{ or } 1\qquad j=1, 2, 3.$$

The linear programming optimum is $\bar{x}=(1/3,\ 1/3,\ 1/3)$ and the halflines originating at $\bar{x}$ are $\xi_1=\{x|x=\bar{x}-\bar{a},\ \lambda,\ \lambda\geq 0\}$ and $\xi_2=\{x|x=\bar{x}-\bar{a}_2\lambda,\ \lambda\geq 0\}$ for nonbasic variables $y_1$ and $y_2$, respectively (the slacks), where $\bar{a}_1=(-1/3,\ -1/3,\ 2/3)$ and $\bar{a}_2=(2/3,\ -1/3,\ -1/3)$. Hence $\xi_1=\{x|x=(1/3+1/3\ \lambda,\ 1/3+1/3\ \lambda,\ 1/3-2/3\ \lambda),\ \lambda\geq 0\}$, and $\xi_2=\{x|x=(1/3-2/3\ \lambda,\ 1/3+1/3\ \lambda,\ 1/3+1/3\ \lambda),\ \lambda\geq 0\}$.

In this example, $Q=\{(x_1,\ x_2,\ x_3)|x_1+x_2+x_3=1,\ 0\leq x_i\leq 1,\ i=1,\ 2,\ 3\}$, $n=m=3$, $k=1$, and $Q^*=Q^*_{m(n)}$.

Halfline $\xi_1$ intersects simultaneously the following two facets of $Q^*$: $\{x\epsilon Q^*|x_1-x_2-x_3=1\}$ and $\{x\epsilon Q^*|-x_1+x_2-x_3=1\}$, for $\lambda_1=2$, since $\pm(1/3+1/3\ \lambda_i)\mp(1/3+1/3\ \lambda_1)-(1/3-2/3\ \lambda_1)=1$ yields $\lambda_1=2$. The facet of S associated with vertex (1, 1, 0) of the unit cube is $x_1+x_2-x_3=(v+2)/(v+1)$ $(k=1,\ p=2)$. We show that for all values of $v>-1/2$, $\xi_1$ intersects this facet before bd $Q^*$. This intersection occurs when $(1/3+1/3\ \lambda)+(1/3+1/3\ \lambda)-(1/3-2/3\ \lambda)=(v+2)/(v+1)$, i.e., when $\lambda=\bar{\lambda}_1=(2v+5)/(4v+4)$, and it is easy to check that $\bar{\lambda}_1<2$ for all values of $v>-1/2$.

By symmetry, $\xi_2\cap$ bd $Q^*$ and $\xi_2\cap$ bd $S$ are defined by $\lambda_2=2$ and $\bar{\lambda}_2\leq(2v+5)/(4v+4)$ $(<2$ for all $v>-1/2)$.

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# SOLVING A (0, 1) HYPERBOLIC PROGRAM BY BRANCH AND BOUND

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**ABSTRACT**

A problem in (0, 1) hyperbolic programming is formulated and solved by the use of branch and bound methods. Computational results are presented including a comparison among several branching rules. Heuristic methods for quickly finding relatively good feasible solutions are presented and tested. The problem finds application in the scheduling of common carriers.

In the solution of the main problem, a subproblem is identified and solved. A geometric analogue is presented, which allows an interesting interpretation of the subproblem. The subproblem itself finds application in the design of gambles.

## 1. INTRODUCTION

The problem that this paper solves is:

$$\max F(x) = \sum_{i=1}^{S} \frac{\sum_{j=1}^{T} a_{ij}x_j}{\sum_{j=1}^{T} b_{ij}x_j},$$

subject to

(1) $$\sum_{j=1}^{T} x_j = n$$

and

$$x_j = 0, 1 \text{ for all } j,$$

where $x = (x_1, \ldots, x_T)$ and $b_{ij} > 0$ for all $i, j$. Because of the nature of the decision variables and the structure of the objective function the problem might be called a (0, 1) hyperbolic program, (see [1], [2] and [5]).

In preparation for the solution of problem (1), the paper first solves the following subproblem:

$$\max f(x) = \frac{\sum_{j=1}^{T} a_j x_j}{\sum_{j=1}^{T} b_j x_j}$$

subject to

$$\sum_{j=1}^{T} x_j = n, \tag{2}$$

$$x_j = 1 \quad \text{for } j \epsilon \Psi,$$

$$x_j = 0 \quad \text{for } j \epsilon \Phi,$$

and

$$x_j = 0, 1 \quad \text{for all } j,$$

where $b_j > 0$ for all $j$ and where $\Psi$ and $\Phi$ are disjoint subsets of $\{1, 2, \ldots, T\}$ satisfying $\|\Psi\| \leq n$ and $\|\Phi\| \leq T - n$. A geometric analogue for problem (2) will allow an interesting interpretation for its optimal solution.

A branch and bound algorithm will then be presented for the optimal solution of problem (1). Computational results will be reported along with a comparison of several branching rules. Heuristic methods for quickly finding good feasible solutions to problem (1) will also be discussed.

An application of problem (2) to the design of gambles and of problem (1) to the scheduling of common carriers completes the paper.

## 2. THE SUBPROBLEM

In this section an algorithm for problem (2) will be derived. Consider first the case of $\Psi = \Phi = \{\phi\}$, called the "free subproblem of order $n$":

$$\max f(x) = \frac{\sum_{j=1}^{T} a_j x_j}{\sum_{j=1}^{T} b_j x_j}$$

subject to

$$\sum_{j=1}^{T} x_j = n \tag{3}$$

and

$$x_j = 0, 1 \quad \text{for all } j,$$

where $b_j > 0$ for all $j$. The following will develop a geometric analogue for this problem.

Each pair $(b_j, a_j)$ may be identified as a point in the real half-plane $R_+^2 = \{(x, y) | x > 0, -\infty < y < \infty\}$. The ratio $a_j/b_j$ may be interpreted as the slope of the line through the origin and the point $(b_j, a_j)$. Vector addition of any two points $(b_j, a_j)$ and $(b_k, a_k)$ produces the resultant $(b_j + b_k, a_j + a_k)$ with slope $(a_j + a_k)/(b_j + b_k)$. Therefore the free subproblem of order $n$ has the following geometric interpretation: given $T$ vectors in $R_+^2$, find a subset of $n$ of those vectors with the property that when added vectorially, the resultant has maximum slope. Exhibit 1 illustrates this geometric analogue.

One might initially speculate that this problem could be solved by selecting those $n$ vectors with the largest slopes. However, this is not the case, as the counter-example of Exhibit 2 shows. One can

prove, though, that the vector with the largest slope is in the optimal solution. This theorem is easily proved using the following lemma which proves that addition of vectors with unequal slopes produces a resultant with an intermediate slope.

LEMMA: Assume $a_k/b_k > a_l/b_l$ with $b_k, b_l > 0$. Then

$$\frac{a_k}{b_k} > \frac{a_k + a_l}{b_k + b_l} > \frac{a_l}{b_l}.$$

PROOF: For the first inequality,

$$\frac{a_k}{b_k} - \frac{a_k + a_l}{b_k + b_l} = \frac{a_k b_l - a_l b_k}{b_k(b_k + b_l)} > 0.$$

Similarly, for the second inequality,

$$\frac{a_k + a_l}{b_k + b_l} - \frac{a_l}{b_l} = \frac{a_k b_l - a_l b_k}{b_l(b_k + b_l)} > 0.$$

End of proof.

THEOREM: If, for problem (3)

$$\frac{a_k}{b_k} > \frac{a_j}{b_j} \qquad \text{for all } j,$$

then, $x_k = 1$ in the optimal solution.

PROOF: (By contradiction) Assume the optimal solution to problem (3) is $x_j = 1$ for all $j \in S$, and assume $k \notin S$. Find the smallest of the ratios in the optimal set; that is, find $l \in S$ such that

$$\frac{a_l}{b_l} \leqq \frac{a_j}{b_j} \qquad \text{for all } j \in S.$$

To avoid triviality, assume $a_k/b_k > a_l/b_l$. It will be shown that replacing $x_l = 1$ in the optimal solution by $x_k = 1$ yields a better solution, which is a contradiction. Let $S' = S - \{l\}$. Then,

$$\frac{a_k}{b_k} > \frac{\sum_{j \in S'} a_j}{\sum_{j \in S'} b_j} > \frac{a_l}{b_l}$$

from the lemma. (Note that there may be one equality in the above expression.) It follows from the lemma that

$$\frac{a_k + \sum_{j \in S'} a_j}{b_k + \sum_{j \in S'} b_j} > \frac{\sum_{j \in S} a_j}{\sum_{j \in S} b_j},$$

which is a contradiction. □

With $k$ defined as in the statement of the theorem, it is seen that problem (3) is equivalent to

$$\max f(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_T) = \frac{a_k + \sum_{j \neq k} a_j x_j}{b_k + \sum_{j \neq k} b_j x_j}$$

subject to

$$\sum_{j \neq k} x_j = n-1 \tag{4}$$

and

$$x_j \ldots 0, 1 \qquad \text{for all } j (j \neq k).$$

Problem (4) may now be transformed into a free subproblem of order $n-1$ by adding

$$\frac{a_k}{n-1}\left[\sum_{j \neq k} x_j - (n-1)\right]$$

to the numerator of the objective function and

$$\frac{b_k}{n-1}\left[\sum_{j \neq k} x_j - (n-1)\right]$$

to the denominator. The result is:

$$\max f(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_T) = \frac{\sum_{j \neq k}\left[a_j + \frac{a_k}{n-1}\right] x_j}{\sum_{j \neq k}\left[b_j + \frac{b_k}{n-1}\right] x_j}$$

subject to

$$\sum_{j \neq k} x_j = n-1 \tag{5}$$

and

$$x_j = 0, 1 \qquad \text{for all } j (j \neq k).$$

The theorem is again applicable to problem (5). It follows that the free subproblem or order $n$ may be solved in just $n$ steps. At each of the first $n-1$ steps the variable with the biggest ratio is set to 1 and then the problem is transformed to a lower order free problem. At the $n$th step, the variable with the biggest ratio is set equal to 1, and that ratio is the optimal value of the objective function.

To complete the geometric analogy, one can think of the transformation operation as a vector ad-

dition of the appropriate fraction, $1/(n-1)$, of the vector with the largest slope to all of the remaining vectors to ensure that when $n-1$ of the remaining vectors are selected and vectorially added, the one with the largest slope gets completely reconstructed and contributes to the resultant. Exhibit 3 illustrates this idea.

It is clear that this method may be applied to the more general subproblem, problem (2). In this case the first step would be an imposition on the problem of $x_j=1$ for $j\epsilon\Psi$, and $x_j=0$ for $j\epsilon\Phi$, followed by a transformation to a free subproblem of order $n-\|\Psi\|$. In all, $n-\|\Phi\|+1$ iterations would be required. The execution time for an algorithm based upon this method will be approximately proportional to $n'[T'-(n'+1)/2]$ where $n'=n-\|\Psi\|, T'=T-\|\Phi\|$. Exhibit 4 presents a numerical example.

An application for this subproblem will be given in Section 5. This section concludes with a mention of several related problems. Consider

$$\max f(x)=\frac{a_0+\sum_{j=1}^{T} a_jx_j}{b_0+\sum_{j=1}^{T} b_jx_j} \tag{6}$$

subject to

$$x_j=0,\ 1 \qquad \text{for all } j,$$

where $b_j>0$ for all $j$. If the simple summation constraint

$$\sum_{j=1}^{T} x_j=n$$

is added to problem (6), then it can be solved by the methods of this section. Without the summation constraint and if $a_0=b_0=0$, the optimal solution follows immediately from the theorem, and is $x_k=1$ for any or all $k$ such that

$$\frac{a_k}{b_k}\geq\frac{a_j}{b_j} \qquad \text{for all } j.$$

With nonzero $a_0$ and $b_0$, the solution to (6) is not as immediate. Observe, first, that (6) may be transformed to an equivalent problem where all the numerator coefficients are positive. This follows because

$$f(x)=\frac{\Sigma a_jx_j}{\Sigma b_jx_j}=\frac{\Sigma(a_j+lb_j)x_j}{\Sigma b_jx_j}-l.$$

Since $b_j>0$, one can always choose $l$ sufficiently large to make $a_j+lb_j>0$ for all $j$. Thus problem (6) may be solved in general by transforming $f(\cdot)$ to $g(\cdot)$ with positive numerator coefficients and then minimizing $g^{-1}$ using Robilliard's unconstrained algorithm [5]. When, with or without the simple summation constraint, additional constraints are added to (6) in the form of inequalities on nondecreasing pseudo-boolean functions, the solution may be found by minimizing $g^{-1}$ using Robilliard's branch and bound algorithm [5].

## 3. THE BRANCH AND BOUND ALGORITHM

This section presents a solution for problem (1). A standard branch and bound strategy is used to search a rooted binary tree. Each node of the tree has associated with it two sets: $\Psi$ contains the indices of those variables which are fixed to 1 at that node; and, $\Phi$ contains the indices of those variables which are fixed to 0 at that node. (At any node $\Psi \cap \Phi = \{\phi\}$ with $\|\Psi\| \leq n$ and $\|\Phi\| \leq T-n$.) Thus each node represents a subset of feasible solutions to (1), that subset completely determined by the corresponding $\Psi$ and $\Phi$ and comprised of all feasible $x$ satisfying $x_j=1$ for $j\epsilon\Psi$ and $x_j=0$ for $j\epsilon\Phi$. Also associated with each node is an upper bound on the values that the objective function, $F$, takes over all the feasible solutions in its corresponding subset, denoted $\overline{F}(x\,|\,\Psi, \Phi)$. A sample tree is given in Exhibit 5.

The branch and bound strategy has been well documented elsewhere (see, for example, [3] and [4]) and will not be presented here. To completely determine the current algorithm there remains to specify the upper bound and the branching rule.

The objective function of (1) may be written as follows:

$$F(x) = \sum_{i=1}^{S} f_i(x),$$

where

$$f_i(x) = \frac{\sum_{j=1}^{T} a_{ij}x_j}{\sum_{j=1}^{T} b_{ij}x_j} \qquad i=1, 2, \ldots, S.$$

It is seen that an upper bound on $F$ over the subset of feasible solutions characterized by $\Psi$ and $\Phi$ can be found from

$$\overline{F}(x|\Psi, \Phi) = \sum_{i=1}^{S} f_i^* (x|\Psi, \Phi), \tag{7}$$

where $f_i^* (x|\Psi, \Phi)$ is the optimal value of the objective function in the subproblem

$$\max f_i^* = \frac{\sum_{j=1}^{T} a_{ij}x_j}{\sum_{j=1}^{T} b_{ij}x_j}$$

subject to

$$\sum_{j=1}^{T} x_j = n \tag{8}$$

$$x_j = 1 \qquad \text{for } j\epsilon\Psi$$

$$x_j = 0 \qquad \text{for } j\epsilon\Phi$$

and

$$x_j = 0, 1 \qquad \text{for all } j.$$

Of course, the $f_i^*$ may be computed by using the methods of section 2. So, to compute $\bar{F}(x|\Psi, \Phi)$ as defined by (7) one must solve $S$ subproblems of type (8). A sample upper bound calculation is given in Exhibit 6.

The branching rule for this problem will be a two-part rule consisting first of the next-node decision (a choice of the node from which to branch) and then of the next-variable decision (a choice of an uncommitted variable at that node to use in branching; that is, a variable to fix to 1 and then to 0 in generating two new nodes.)

The current algorithm makes the next-node decision by picking the node with the largest bound. All other things fixed, this rule ensures that the fewest nodes are ultimately constructed.

Rules for the next-variable decision may be broken into two classes. The first, which might be termed "master-list" rules, requires the prespecification of a fixed ordering of the variables. Then, at a node characterized by $\Psi$ and $\Phi$, the $(\|\Psi\| + \|\Phi\| + 1)$st variable on the list is chosen next. This kind of rule trades less computation at each node for, in general, more nodes in the tree. The alternative to a master-list rule is a "position-dependent" rule (like that of Little, et al., [3]) which picks from among the uncommitted variables at the selected node one which satisfies a node-dependent criterion. Position-dependent rules require more computation at each node, but typically generate fewer nodes.

In the computational results which follow, three different next-variable rules were tested:

**(a) The *f*-Sum Master-List Rule**

This heuristic for ordering the master-list has proven to be fairly effective. Its rationale is as follows: if any of the $f_i^*(x|\{\phi\}, \{\phi\})$ † are considerably larger than the rest, variables which take the value 1 in the optimal solution of these "dominant" subproblems are likely to take the value 1 in the optimal solution to (1). To generate the list, $m(1), \ldots, m(T)$, first solve for $f_i^*(x|\{\phi\}, \{\phi\})$, $i = 1, 2, \ldots, S$. Note that this must be done anyway to compute the first bound in the solution of (1). Now define

$$Q_j = \sum_i f_i^*(x|\{\phi\}, \{\phi\}) \qquad j = 1, 2, \ldots, T$$

where the summation runs over those subproblems for which $x_j = 1$. Finally, order the master-list so that

$$Q_{m(i)} \geq Q_{m(i+1)} \qquad i = 1, 2, \ldots, T-1. \tag{9}$$

**(b) The Position-Dependent *f*-Sum Rule**

This rule is the position-dependent version of the preceding rule. When branching from the node characterized by $\Psi$ and $\Phi$, there are $T - \|\Psi\| - \|\Phi\|$ uncommitted variables among which to choose. One can compute a (position-dependent) *f*-sum, $Q_j$, for each uncommitted $x_j$ using

† Note that $\{\phi\}$ stands for the empty set.

$$Q_j = \sum_i f_i^*(x|\Psi, \Phi) \qquad \text{for all } j \text{ where } x_j \text{ is uncommitted}$$

where the summation runs over all $i$ for which $x_j = 1$. Choose variable $x_m$ next, where

$$Q_m \geq Q_j \qquad \text{for all } j \text{ where } x_j \text{ is uncommitted.}$$

**(c) The Position-Dependent Lex Rule**

The rationale here is that the order in which variables are set equal to 1 in the optimal solution of subproblems is a measure of the strength of their contribution to those solutions. At a node characterized by $\Psi$ and $\Phi$, when computing $f_i^*(x|\Psi, \Phi)$ where $i = 1, 2, \ldots, S$, define

$$l_{j;k,i} = \begin{cases} 1 & \text{if } x_j \text{ is the } k\text{th (uncommitted) variable set to 1 in the solution of the } i\text{th subproblem} \\ 0 & \text{otherwise} \end{cases}$$

$i = 1, 2, \ldots, S, k = 1, 2, \ldots, n - \|\Psi\|$, for all $j$ where $x_j$ is uncommitted.

Now let

$$L_{j,k} = \sum_{i=1}^{S} l_{j;k,i} \qquad j \text{ and } k \text{ as above}$$

$L_{j,k}$ represents the number of times variable $x_j$ was the $k$th variable to be set to 1. One would pick $x_m$ as the next variable where, letting $n' = n - \|\Psi\|$,

$$(L_{m1}, \ldots, L_{mn'}) \underset{L}{\geq} (L_{j1}, \ldots, L_{jn'}) \qquad \text{for all } j \text{ with } x \text{ uncommitted}$$

where $\underset{L}{\geq}$ denotes lexicographic comparison.

Exhibits 7, 8, and 9 give examples of the application of these next-variable rules. The branch and bound algorithms of this section were run on several test problems, the results of which are given in Exhibit 10.

Five 10-variable ($T = 10$) test problems and five 15-variable ($T = 15$) test problems were run to compare the three next-variable rules. The first table in Exhibit 10 shows that for these test problems the $f$-sun master-list rule was superior. Even though the two position-dependent rules, in general, generated fewer nodes than the master-list rule, the lower computation times at each node under the master-list rule made it more efficient.

The second table in Exhibit 10 illustrates the dependence of computation times on $n$. Problems 15–6, 15–1, and 15–7 were identical except that $n$ changed from 4 to 7 to 11, respectively. The apparent poor performance of the algorithm for problem 15–7 arises because the time to solve subproblems increases with $n$ (see section 2). If $n > T/2$ one might suspect that transforming the problem using binary complementation; that is, setting $y_j = 1 - x_j$, $j = 1, 2, \ldots, T$, would, by converting the constraint to

$$\sum_{j=1}^{T} y_j = T - n,$$

improve computation times. This will not work, in general, because this transformation can produce negative demoniator coefficients, and thus problem (3) is not closed under binary complementation.

Perhaps the best that can be done when $n > T/2$ is to run through the master-list in reverse order.

## 4. HEURISTICS

Three heuristics for quickly generating (relatively) good feasible solutions to (1) were developed. The results of limited testing of these heuristics will be reported in this section. Such heuristics which operate very quickly are useful in that they provide a "feel" for the optimal solution to (1). Also many optimum seeking algorithms are made more efficient if a relatively good feasible solution is available early. (This is the case, for example, for branch and bound algorithms if the next node is not selected to be the active node with the extremal bound.) The three heuristics are:

### (a) The *f*-Sun Heuristic

Prepare a master-list $m(1), \ldots, m(T)$ according to the *f*-sum master-list rule of section 3. Then set

$$x_{m(i)} = 1 \qquad i = 1, \ldots, n.$$

### (B) The Lex Heuristic

With reference to the position-dependent lex rule of section 3, solve the $S$ free subproblems given by (8), and prepare a lex table

$$L_{j,k} \qquad \begin{matrix} j = 1, 2, \ldots, T \\ k = 1, 2, \ldots, n. \end{matrix}$$

Select those $n$ variables which lexicographically rank 1 through $n$ to take the value 1.

### (c) The Ratio-Sum Heuristic

The rationale for this heuristic, which requires the least computation of the three, is that any variable for which $a_{ij}/b_{ij}$ is exceedingly large for some $i$, has a good chance of taking the value 1 in the optimal solution to (1). Proceed by letting

$$\rho_j = \sum_{i=1}^{S} \frac{a_{ij}}{b_{ij}} \qquad j = 1, \ldots, T.$$

Arrange the $\rho_j$ in nonincreasing order and set $x_j = 1$ if $\rho_j$ ranks from 1 through $n$.

These three heuristics were tested on five problems with $T = S = 10$, $n = 5$. The results are presented in Exhibit 11. The results of this limited testing indicate that the *f*-sum heuristic seems to be best, however, none of the heuristics really does an excellent job.

It is perhaps the structure of problem (3)—a sum of ratios—which limits the potential of quick heuristics and makes the refinement of an optimum-seeking algorithm a more meaningful challenge. To this end the question of a "best" branch and bound algorithm for this problem is being addressed in current work. This involves experimenting with other upper bounds and other branching rules to find the best combination of trade-offs for this type of problem.

## 5. APPLICATIONS

This section first derives an application of the free subproblem of order $n$ to the design of gambles. Let $0_1, \ldots, 0_T$ be possible outcomes of a random experiment. Associate with $0_j$ the scalar reward $r_j$ and the positive scalar "probability weight" $w_j$. The problem is to design a gamble consisting of only $n$ possible outcomes which maximizes the expected reward. The probability weights are used to provide gamble-dependent outcome probabilities in the following sense: for a gamble which is comprised of the $n$ outcomes $0_j$, $j \epsilon S$, the probability of $0_j$ attaining is given by

$$p(0_j) = \frac{w_j}{\sum_{k \epsilon S} w_k} \qquad \text{for all } j \epsilon S.$$

To formulate, let $x_j$ equal 0 or 1 depending, respectively, upon whether $0_j$ is excluded from or included in the gamble. The design problem is thus:

$$\max f(x) = \frac{\sum_{j=1}^{T} r_j w_j x_j}{\sum_{j=1}^{T} w_j x_j}$$

subject to

$$\sum_{j=1}^{T} x_j = n$$

and

$$x_j = 0, 1 \qquad \text{for all } j.$$

This is a free subproblem of order $n$.

The motivation for tackling problem (1) was an application to the scheduling of common carriers. Given a common carrier link between points A and B, and given a *population of potential travellers* from A to B, *for this carrier*, Shaw [6] has developed a methodology for determining the proportion, $Q$, of that potential travelling population which will actually use any particular schedule of trips from A to B by the carrier. He has thus quantified the effect of *schedule quality* on the demand for transportation.

A natural question which arises is: given that $n$ trips are to be made from A to B, how should they be scheduled to maximize $Q$? Shaw has solved this problem quite effectively by using a modified enumerative computational strategy. However, a slight alteration to Shaw's model allows the problem to be formulated as a problem of type (1).

Partition the scheduling period into $T$ time intervals. Let $x_j$ equal 1 or 0 depending, respectively,

upon whether or not a trip is scheduled to depart at time $j$. Let $D_i$ be the probability that a member of the potential travelling population will select time $i$ as his *preferred departure time*. Let $T_{ij} > 0$ be the potential traveller's *disaccommodation* with a departure at time $j$ if his preferred departure time is $i$.

For a given schedule $(x_1, \ldots, x_T)$ assume that a potential traveller with preferred departure time $i$ will actually select the scheduled departure at time $j$ with probability

$$\frac{x_j T_{i,j}^{-\alpha}}{\sum_{k=1}^{T} x_k T_{i,k}^{-\alpha}} \qquad j=1, 2, \ldots, T, \tag{10}$$

where $\alpha > 0$. (This rule for selecting departures is a slight generalization of Shaw's rule.) Finally, assume that $a(T_{ij})$ is the probability that a potential traveller who has selected disaccommodation $T_{ij}$ [by virtue of time $i$ being his desired departure time and time $j$ being his selected departure time via rule (10)] will tolerate the disaccommodation $T_{ij}$ and actually travel on the carrier.

It follows that for any given schedule, $Q$ may be written as

$$Q = \sum_{i=1}^{T} D_i \sum_{j=1}^{T} \frac{T_{ij}^{-\alpha} x_j}{\sum_{k=1}^{T} T_{ik}^{-\alpha} x_k} a(T_{ij}).$$

Thus, an optimal schedule of $n$ trips will be given by:

$$\max Q = \sum_{i=1}^{T} \frac{\sum_{j=1}^{T} D_i a(T_{ij}) T_{ij}^{-\alpha} x_j}{\sum_{j=1}^{T} T_{ij}^{-\alpha} x_j}.$$

subject to

$$\sum_{j=1}^{T} x_j = n$$

and

$$x_j = 0, 1 \qquad \text{for all } j,$$

which is a problem of type (1).

EXHIBIT 1: Geometric Interpretation of a Free Subproblem of Order $n$.

Consider the problem

$$\max f(x) = \frac{6x_1 + 4x_2 + 2x_3 - x_4}{2x_1 + 5x_2 + 5x_3 + x_4}$$

subject to

$$x_1 + x_2 + x_3 + x_4 = 2$$

and

$$x_j = 0, 1 \qquad \text{for all } j.$$

The accompanying graph depicts the vectors (2, 6), (5, 4), (5, 2), and (1, −1). An equivalent statement of the preceding problem is to pick from among these four vectors those two which when added produce a resultant of maximum slope.

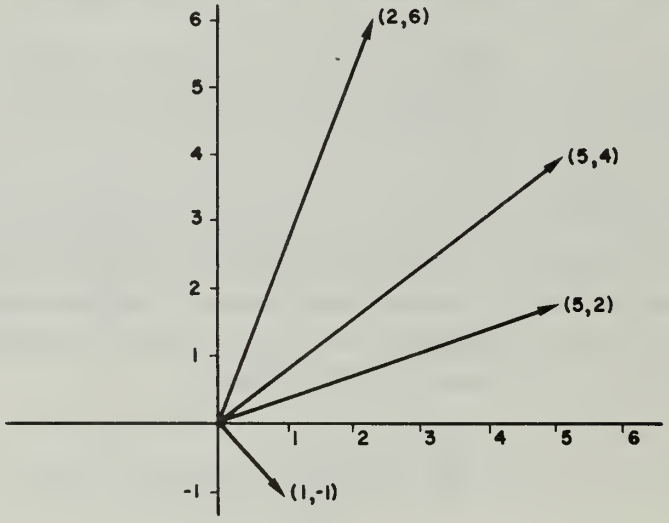



EXHIBIT 2: A Counter-Example.

The optimal solution of the problem in Exhibit 1 is $x_1 = x_4 = 1$ (with $x_2 = x_3 = 0$). The graph compares this solution with another feasible solution, $x_1 = x_2 = 1$ (with $x_3 = x_4 = 0$).

EXHIBIT 3: Geometric Interpretation of the Transformation Operation.

Consider the problem

$$\max f(x) = \frac{8x_1 + 7x_2 + 2x_3 - 2x_4 - 5x_5}{2x_1 + 7x_2 + 3x_3 + 3x_4 + 4x_5},$$

subject to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 3$$

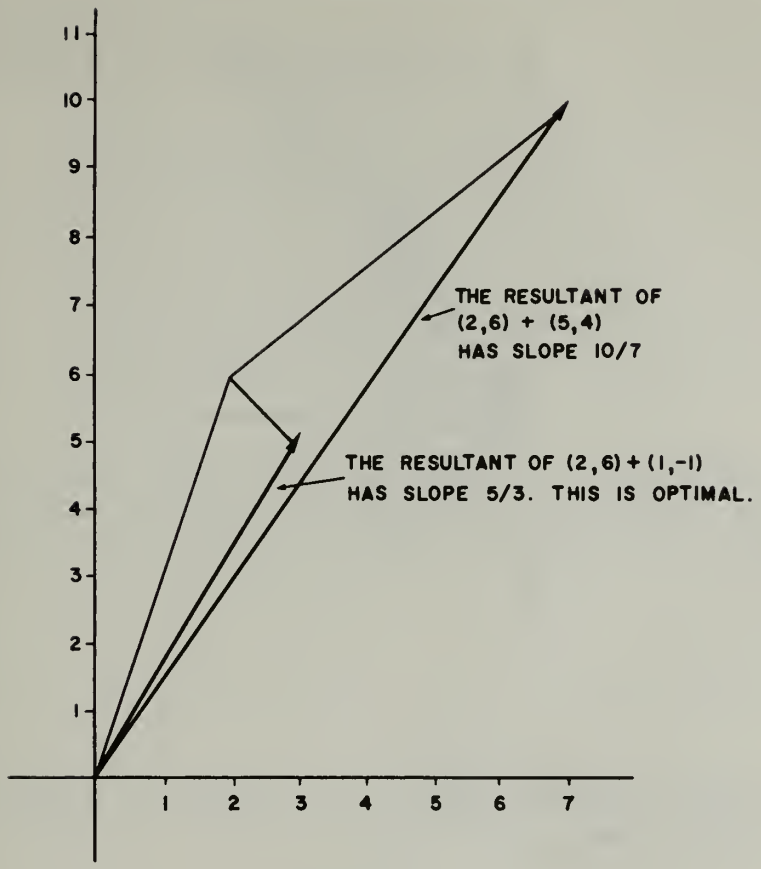
and

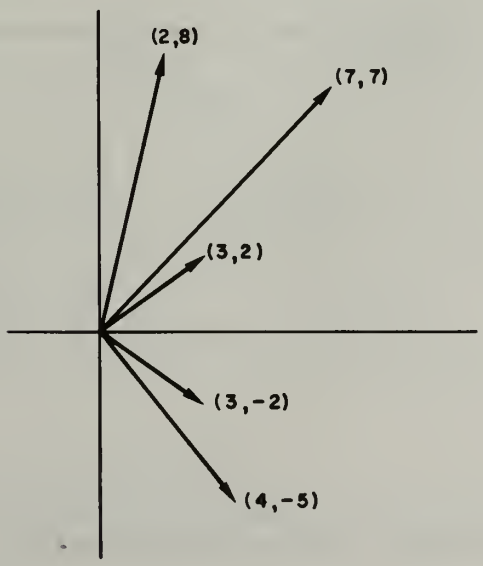
$$x_j = 0, 1 \qquad \text{for all } j.$$

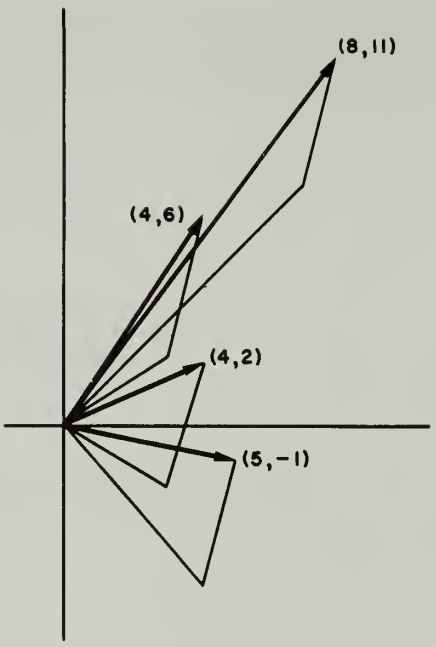
Graph 1 depicts the five original vectors.

At step 1, since (2, 8) has the largest slope, set $x_1 = 1$. Now transform by adding (1, 4) to the remaining vectors, as shown in Graph 2. It is clear that at step 2, one must set $x_3 = 1$, and so on. The transformation ensures that when the best two vectors of Graph 2 are selected, (1, 4) will be included twice, and thus (2, 8) will contribute to the ultimate resultant.

EXHIBIT 4: A Numerical Example of the Solution of Subproblem (2).


11
10
9
8
7
6
5
4
3
2
1
1
2
3
4
5
6
7
THE RESULTANT OF
(2,6) + (5,4)
HAS SLOPE 10/7
THE RESULTANT OF (2,6) + (1,-1)
HAS SLOPE 5/3. THIS IS OPTIMAL.


(2,8)
(7,7)
(3,2)
(3,-2)
(4,-5)




Consider the problem of type (2) with $T=9$, $n=4$, $\Psi=\{5\}$, $\Phi=\{7, 8\}$, and the $a_j$ and $b_j$ as given by the first two rows of the table:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | $a_k b_k$ | Transformed order | Variables set to 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_j$ | 1 | 3 | 9 | 6 | 3 | 7 | 9 | 4 | 2 | 3 | 3 | 5 |
| $b_j$ | 7 | 1 | 3 | 2 | 9 | 5 | 2 | 1 | 4 | 9 | | |
| $a_j$ | 2 | 4 | 10 | 7 | | 8 | | | 3 | 10 | 2 | 3 |
| $b_j$ | 10 | 4 | 6 | 5 | | 8 | | | 7 | 6 | | |
| $a_j$ | 7 | 9 | | 12 | | 13 | | | 8 | 12 | 1 | 4 |
| $b_j$ | 13 | 7 | | 8 | | 11 | | | 10 | 8 | | |
| $a_j$ | 19 | 21 | | | | 25 | | | 20 | 21 | | 2 |
| $b_j$ | 21 | 15 | | | | 19 | | | 18 | 15 | | |

Here the optimal solution is $x_2=x_3=x_4=x_5=1$ for a value of 21/15.

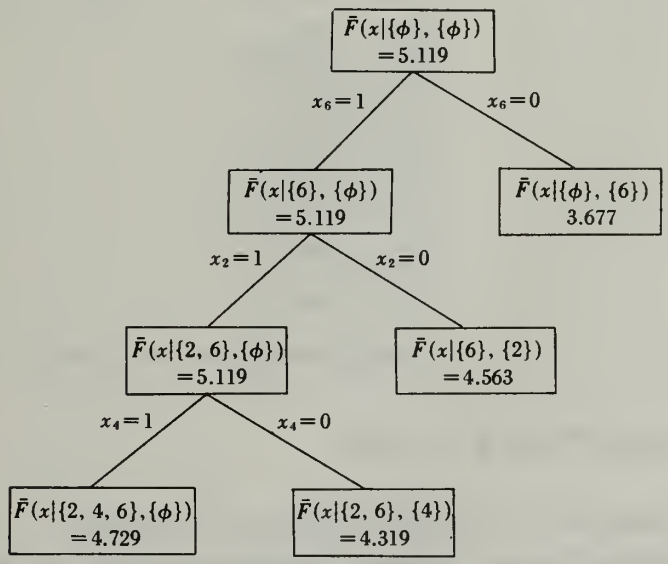
EXHIBIT 5: A Sample Tree.

Consider the problem of type (1) with $T=6$, $S=3$, $n=3$, and

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $a_{1j}$ | 9 | 2 | 3 | 5 | 8 | 7 |
| $b_{1j}$ | 6 | 2 | 8 | 2 | 9 | 1 |
| $a_{2j}$ | 9 | 8 | 5 | 2 | 1 | 3 |
| $b_{2j}$ | 8 | 8 | 7 | 3 | 6 | 1 |
| $a_{3j}$ | 9 | 2 | 2 | 4 | 3 | 5 |
| $b_{3j}$ | 7 | 2 | 7 | 6 | 4 | 5 |

The tree for the solution of this problem was generated by the branch and bound algorithm of Section 3. The next-variable decision was made by using the position-dependent $f$-sum rule. Note that $\bar{F}(x|\Psi, \Phi)$ denotes an upper bound on the value of $F$ over all feasible solutions $x$ with $x_j=1, j\epsilon\Psi$ and $x_j=0, j\epsilon\Phi$.

TREE:




Optimal
$x_2=x_4=x_6=1$
$F^*=4.729$

$\binom{6}{3}=20$

$2\binom{6}{3}-1=39$

EXHIBIT 6: A Sample Upper Bound Calculation.

For the sample problem of Exhibit 5, the computation of $\bar{F}(x|\{6\}, \{2\})$ proceeds as follows:

1. The values of $f_i^*(x|\{6\}, \{2\})$, $i=1, 2, 3$, are found by using the method of section 2. These are:

| $i$ | $f_i^*(x\|\{6\}, \{2\})$ |
|---|---|
| 1 | 2.333 |
| 2 | 1.167 |
| 3 | 1.063 |

2. These are summed, yielding $\bar{F}(x|\{6\}, \{2\}) = 4.563$.

EXHIBIT 7: Constructing the $f$-Sum Master-List.

For the sample problem of Exhibit 5 the optimal solutions of the three free subproblems are tabulated below:

| $i$ | $f_i^*(x\|\{\phi\}, \{\phi\})$ | Solution |
|---|---|---|
| 1 | 2.800 | $x_2 = x_4 = x_6 = 1$ |
| 2 | 1.176 | $x_1 = x_2 = x_6 = 1$ |
| 3 | 1.143 | $x_1 = x_2 = x_6 = 1$ |

Thus the $f$-sums are

| $j$ | $Q_j$ |
|---|---|
| 1 | 2.319 |
| 2 | 5.119 |
| 3 | 0.000 |
| 4 | 2.800 |
| 5 | 0.000 |
| 6 | 5.119 |

So, one master-list satisfying (9) is (6, 2, 4, 1, 3, 5).

EXHIBIT 8: An Application of the Position-Dependent $f$-Sum Rule.

For the sample problem of Exhibit 5, consider the selection of the next variable at the node $\Psi = \{2, 6\}$, $\Phi = \{\phi\}$. The following table may be prepared during the computation of $\bar{F}(x|\{2, 6,\}, \{\phi\})$:

| $i$ | $f_i^*(x\|\{2, 6\}, \{\phi\})$ | Solution |
|---|---|---|
| 1 | 2.800 | $x_2 = x_4 = x_6 = 1$ |
| 2 | 1.176 | $x_1 = x_2 = x_6 = 1$ |
| 3 | 1.143 | $x_1 = x_2 = x_6 = 1$ |

This allows the computation of the (position-dependent) $f$-sums for the uncommitted variables:

| $j$ | $Q_j$ |
|---|---|
| 1 | 2.319 |
| 3 | 0.000 |
| 4 | 2.800 |
| 5 | 0.000 |

Thus, variable $x_4$ is chosen next.

EXHIBIT 9: An Application of the Position-Dependent Lex Rule.

The next-variable decisions for the sample problem of Exhibit 5 were not made by using the position-dependent lex rule; however, this rule might have been employed at, say, the node $\Psi=\{6\}$, $\Phi=\{\phi\}$.

The following table records the optimal solutions found in the computation of $f_i^*(x|\{6\}, \{\phi\})$, $i=1, 2, 3$. Note that variable $x_6$ is forced into each optimal set, and the other variables are recorded in the order in which they entered.

| $i$ | Solution |
|---|---|
| 1 | $x_6=x_4=x_2=1$ |
| 2 | $x_6=x_1=x_2=1$ |
| 3 | $x_6=x_1=x_2=1$ |

A (position-dependent) lex table may now be constructed:

| $L_{j,k}$ | | $k$ = 1 | 2 |
|---|---|---|---|
| | 1 | 2 | 0 |
| | 2 | 0 | 3 |
| | 3 | 0 | 0 |
| j: | 4 | 1 | 0 |
| | 5 | 0 | 0 |

This rule would have selected $x_1$ as the next variable.

EXHIBIT 10: Computational Results.

For each of the test problems the numerator and denominator coefficients were randomly selected integers uniformly distributed between 1 and 15. Execution times are in seconds and are precise to

within an estimated ±5 percent. Also recorded is the number of nodes generated in determining each optimal solution. To help interpret these numbers, note that the number of nodes in the complete tree implicitly defined by the branch and bound algorithm is $2\binom{T}{n}-1$.

Comparison of Next-Variable Rules

| Problem | | *f*-sum master list | | Position-dependent *f*-sum | | Position-dependent Lex | |
|---|---|---|---|---|---|---|---|
| | | Execution time | Nodes | Execution time | Nodes | Execution time | Nodes |
| $T=S=10$ | 10–1 | 4.6 | 81 | 5.4 | 79 | 5.0 | 65 |
| $n=5$ | 10–2 | 3.4 | 73 | 5.2 | 67 | 4.8 | 59 |
| $2\binom{10}{5}-1$ | 10–3 | 7.2 | 135 | 8.4 | 131 | 7.7 | 117 |
| | 10–4 | 9.2 | 177 | 11.2 | 203 | 9.6 | 153 |
| $=503$ | 10–5 | 6.1 | 117 | 8.2 | 127 | 6.5 | 105 |
| $T=S=15$ | 15–1 | 248 | 2157 | 301 | 2453 | 264 | 1989 |
| $n=7$ | 15–2 | 236 | 1991 | 336 | 2747 | 299 | 2321 |
| $2\binom{15}{7}-1$ | 15–3 | 258 | 2165 | 291 | 2387 | 255 | 1993 |
| | 15–4 | 266 | 2321 | 319 | 2559 | 276 | 2137 |
| $=17{,}869$ | 15–5 | 260 | 2343 | 291 | 2289 | 253 | 1909 |

The Effect of Varying *n*

| Problem | | *n* | Execution times* | Nodes |
|---|---|---|---|---|
| $T=S=10$ | 15–6 | 4 | 59.9 | 705 |
| | 15–1 | 7 | 248 | 2156 |
| | 15–7 | 11 | 101 | 705 |
| $2\binom{15}{4}-1=2\binom{15}{7}-1=2729$ | | | | |

*next-variable from *f*-sum master-list

EXHIBIT 11; Heuristic Computational Results

Note that computation time for each of the heuristics was a fraction of a second. For each problem $\mu$ is an estimate of the mean value of all feasible solutions and $\sigma^2$ is an estimate of the corresponding variance. (Histograms of all feasible solutions for these problems are roughly unimodal and slightly skewed to the right.)

| | 10–1 | | 10–2 | | 10–3 | | 10–4 | | 10–5 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Solution | Value | Solution | Value | Solution | Value | Solution | Value | Solution | Value |
| Optimal | (0000110111) | 14.79 | (0100011101) | 14.48 | (0011101001) | 12.17 | (1101000011) | 11.39 | (0011010110) | 13.22 |
| Lex | (0100111010) | 12.39 (17) | (0101011001) | 12.24 (6) | (0001111001) | 11.75 (4) | (0101011001) | 10.07 (20) | (0100010111) | 10.94 (27) |
| *f*-sum | (1000110111) | 14.79 (0) | (0001011101) | 12.43 (5) | (0011111000) | 11.04 (10) | (1100010110) | 10.37 (15) | (1011010100) | 11.97 (6) |
| Ratio sum | (0100110011) | 13.56 (5) | (1101010001) | 11.21 (20) | (0101101001) | 11.33 (7) | (0111010001) | 9.71 (33) | (0111011000) | 11.84 (9) |
| | $\mu = 11.1$ $\sigma^2 = 1.25$ | | $\mu = 10.5$ $\sigma^2 = 1.36$ | | $\mu = 10.1$ $\sigma^2 = 0.53$ | | $\mu = 9.4$ $\sigma^2 = 0.45$ | | $\mu = 10.9$ $\sigma^2 = 1.60$ | |

The numbers in brackets under the heuristic values are estimates of the percentage of feasible solutions which have larger values.

## ACKNOWLEDGEMENT

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# EXTENSIONS AND REDUCTIONS IN THE VON NEUMANN MODEL OF AN EXPANDING ECONOMY

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**ABSTRACT**

This paper introduces an extension of the v. Neumann model of an expanding economy. In addition to the conventional nonnegative input and output matrices $A_1$, $B_1$ representing technology, two matrices $A_2$, $B_2$ represent socio-political evaluations and show that there exist solutions to the 4-matrix model. The proof is based on an extension of a constructive proof given by O. Morgenstern and G. L. Thompson. It is shown that this proof is valid only under an additional assumption. The transformation of v. Neumann models (taking consumption into account) into 1 or 2 games is shown and adds an additional condition to M. Morishima's model to guarantee a solution. The equivalence of the v. Neumann model to a maximization problem under a (efficiency) constraint is presented. It is shown that E. Malinvaud's maximality and efficiency criterion—if based on the same assumptions (model)—are equivalent and specify the assumptions which will make the MT-model efficient. The economic evaluation is considered to be of utmost importance.

## 1. THE MT-MODEL

In contrast to "An Open Expanding Economic Model" [26], an earlier extension of the v. Neumann 33] model, the KMT [7] model presented by the same authors O. Morgenstern and G. L. Thompson in 'Private and Public Consumption and Savings in the v. Neumann Model of an Expanding Economy" subsequently referred to as MT) has so far not received a significant amount of attention nor has it been followed up.

This is quite surprising, since the MT-approach—as will be shown—not only constitutes a generalization of previous papers on consumption in v. Neumann Models [3, 11; 13, 14, 21; 27], but may also be considered the initial point of departure for the explanation and clarification of existing problems.

One of the features of the MT-model which I consider to be of utmost importance is the possibility to include utility. In economic theory this is done quite seldom today. Or it is done in an implicit manner.

In addition to the nonnegative Input-matrix $A$, Output-matrix $B$ both $(m \times n)$, fulfills the KMT condition

(**) $$A_i \geq 0, \qquad B^j \geq 0 \qquad i=1, \ldots, m \qquad j=1, \ldots, n$$

subscript $i$ denotes a row, superscript $j$ denotes a column vector). O. Morgenstern and G. L. Thompson introduced (an additional) nonnegative $(m \times n)$ matrix $W$ whose entries $w_{ij}$ represent the amount of good $j$ obtained in the process $i$, if that industry is run at unit intensity. It is of no importance whether that amount of the good constitutes an earning for employers or employees or for both. One decisive step—in contrast to the approach of E. Malinvaud [13, 14, 21] and M. Morishima [27, 31]—takes into account that in addition to introducing income the earnings may be spent or saved. It is assumed

that there are two nonnegative numbers characterizing consumption and saving, respectively, satisfying $0 \leqq c \leqq 1, 0 \leqq s \leqq 1$ which may be interpreted as the propensity to consume or to save.

The components $c\mathbb{W}$ denote the amount of goods consumed, $s\mathbb{W}$ the amount of goods saved, if the industries concerned are run at unit intensity. $\mathbb{W}y$ ($y$ is the normalized $(1 \times n)$ price vector, $x$ is the $(m \times 1)$ normalized vector of intensities) shows the gross cost of labor in the different industries, also on the assumption that they are run at unit intensity.

To preserve the composition of the consumption and saving vectors it is assumed that the income elasticity of demand is 1 (Engel elasticity).

It is obvious that the demand functions of the economy are homogeneous of degree zero with respect to income $(x\mathbb{W}y)$ and prices $(y)$; consequently the economic agents are not subject to money illusion.

Denote $d_j$ as the demand function for good $j$

$$d_j = d_j(y)(x\mathbb{W})y$$

$d_j(y)$ is the proportion of the demand for good $j$, taking into consideration the total income of the economy $x\mathbb{W}y$. Since $d = (d_j)$ characterizes the demand for any good in the corresponding period, it follows that the inner product of the demand vector and the price vector is

$$dy = \sum_j d_j(y) y_j (x\mathbb{W})y.$$

The Engel elasticity of one guarantees that the marginal propensity to consume will be constant; $0 < c \leqq 1$

$$\sum_{j=1}^{n} d_j(y) y_j = c$$

$$dy = c(x\mathbb{W})y.$$

As a result of the above equation the "conservation condition" of the v. Neumann Model may be written in the following way ($\alpha$ denotes the growth factor)

$$x(B - \alpha A) \geqq \alpha x c \mathbb{W}. \tag{1.1.}$$

A part of the income generated by the processes will be saved—and invested, in accordance with the classical assumption of the identity of saving and investment—not in the more tautological interpretation of J. M. Keynes. The reinvested amount $s\mathbb{W}y$ will yield interest and the "profitless condition" of the v. Neumann Model may be written in the following way ($\beta$ denotes the interest factor):

$$(B - \beta A) \leqq s\mathbb{W}y \tag{1.2}$$

From the equations fulfilling the "rule of free goods" and the "rule that inefficient processes must be used with zero intensity" it follows that:

(1.3) $$x(B-\alpha(A+cW))y=0$$

(1.4) $$x(B-\beta(A+sW))y=0.$$

The condition guaranteeing economic solutions (8) is preserved:

(1.5) $$xBy>0.$$

Since the KMT condition does not prevent the economy from breaking up into different sub-economies—or vice versa: it allows the breaking up of the economy—and O. Morgenstern, G. L. Thompson were mainly interested in the economic implications of an economy that is either held together by technical requirements (original v. Neumann assumption) or at least by the tastes of the consumers—it is assumed that:

(1.6) $$A+B+W>0.$$

This assumption does not guarantee the existence of a unique solution because it is possible for $c$ or $s$ to take on the value 0. For this reason (1.6) should be written in the following way:

(1.6') $$A+B+sW>0$$

$$A+B+cW>0.$$

It should be mentioned that an economy may also be held together by governmental activities, denoted by a nonnegative $G(m\times n)$ matrix, so that

$$A+B+W+G>0.$$

As will be seen later this does not influence the structure of the proof.

LEMMA 1: If there are solutions to (1.1–1.5) then $\alpha>0$, $\beta>0$; $x\geqq 0$, $y\geqq 0$. [23].

(1.7) $$\alpha=\frac{xBy}{x(A+cW)y}$$

(1.8) $$\beta=\frac{xBy}{x(A+sW)y}$$

(1.9) $$\alpha/\beta=\frac{x(A+sW)y}{x(A+cW)y}.$$

PROOF: The above definition can be derived directly from the equations (1.3, 1.4). Since: $xBy-xAy-\alpha cxWy=0$ and $xBy-\beta xAy+\beta sxWy=0$. Since (1.5) $xBy>0$ the denominator will not be zero and $\alpha>0$, $\beta>0$.

LEMMA 2: If there are solutions to (1.1–1.5) then: [24]

(1.10) $$c > s = \beta > \alpha > 0$$

(1.11) $$c < s = \alpha > \beta > 0$$

(1.12) $$c = s = \alpha = \beta > 0.$$

PROOF: In addition to the positivity of $\alpha$, $\beta$ and (1.7, 1.8) we immediately get the above stated results.

(1.10) characterizes an economy with positive time preference: The economic agents save, if the interest rate is higher than the expansion rate. The consumption ratio exceeds the saving ratio.

(1.11) describes an economy with negative time preference. The economic entities prefer consumption in the future to consumption now. They save even if the expansion factor exceeds the interest factor. The saving ratio is higher than the consumption ratio, thus facilitating the transfer of income for the future.

(1.12) There is not time preference. As a consequence the interest factor and expansion factor are equal.

The decisive improvement was the interpretation of the v. Neumann Model as two different matrix games.

(1.13) $$M(\alpha) = B - \alpha(A + c\mathbb{W})$$

(1.14) $$M(\beta) = B - \beta(A + s\mathbb{W}).$$

THEOREM 3: Let $x$ and $\alpha^*$ be solutions to $M(\alpha)$ and $y$ and $\beta^*$ solutions to $M(\beta)$: If the positive components of $x$ correspond to those components of the intensity vector of game $M(\beta)$ and the positive components of $y$ correspond to those components of the price vector of game $M(\alpha)$, then $x$, $y$, $\alpha^*, \beta^*$ are solutions.

PROOF: As a result of the assumptions, (1.5) is fulfilled. Let us define $s_1$, $s_2$, $d_1$, $d_2$:

$$s_1 = s(y) = \{j : y_j > 0, \quad j \epsilon (1, \ldots, n)\}$$

$$s_2 = s(x) = \{i : x_1 > 0, \quad i \epsilon (1, \ldots, m)\}$$

$$d_1 = d(x) = \{j : x a_{*j} = 0, \quad j \epsilon (1, \ldots, n)\}$$

$$d_2 = d(y) = \{i : a_{i*} y = 0, \quad i \epsilon (1, \ldots, m)\}.$$

Since for every randomized finite game $s_1 = d_1$ and $s_2 = d_2$ we can write:

$$s_1(M(\alpha)) = d_1(M(\alpha)) \qquad s_2(M(\alpha)) = d_2(M(\alpha))$$

$$s_1(M(\beta))=d_1(M(\beta)) \qquad s_2(M(\beta))=d_2(M(\beta)).$$

In accordance with the assumption of Theorem 3 we can write

$$s_1(M(\alpha))=s_1(M(\beta))=d_1(M(\alpha))=d_1(M(\beta))$$

$$s_2(M(\alpha))=s_2(M(\beta))=d_2(M(\alpha))=d_2(M(\beta))$$

and (1.1–1.4) are fulfilled.

It is obvious that not every economy fulfills the above mentioned conditions. It is necessary to give a more general theorem and proof.

The constructive proof given by O. Morgenstern-G. L. Thompson requires, as will be shown by a counter example, the following extension of the KMT-condition (**):

(***) $$A_i^{(j)} \geqq 0, \qquad B^j_{(i)} \geqq 0 \qquad i=1, \ldots, m$$

$$j=1, \ldots, n.$$

Condition (***) can also be written as follows:

$$(A+cW)_i^{(j)} \geqq 0, \qquad (A+sW)_i^{(j)} \geqq 0, \qquad B_i^{(j)} \geqq 0.$$

As usual subscript $i$ (not in brackets) denotes a row vector, and a superscript $j$ (not in brackets) a column vector of a matrix. The superscript $j$ in brackets $(j)$ refers to a submatrix obtained by deleting a column $j$ from the original matrix $A$. $A_i^{(j)}>0$ implies that the $i$th row vector of the submatrix $A^{(j)}$ of $A$ (reduced by deleting a column $j$) is semipositive. The subscript $i$ in brackets $(i)$ refers to a submatrix obtained by deleting the $i$th row from the original matrix $B$. $B^j_{(i)}$ refers to the $j$th column of the submatrix $B_{(i)}$. The condition (***) obviously includes (**) if $i, j$ (in brackets) take on the value 0.

THEOREM 3*: For a model (1.1–1.5) fulfilling the conditions (xxx) and (1.6′) there always exist solutions $x \geqq 0$, $y \geqq 0$, $\alpha>0$, $\beta>0$, possibly more than one. (Condition (1.6′) can be removed easily).

PROOF: If $m=n=1$ it is obvious that there exists a solution to the model. In case of $m, n>1$ the algorithm of O. Morgenstern-G. L. Thompson leads to a solution, possibly to more than one.

1. Solve the games (1.13, 1.14) as v. Neumann growth models. If the solution comply with Theorem 3 there exists a solution, as was shown. If not then

2. Add the restriction $x_i=0$, if there exists an element 0 in the intensity vector $x$ of game $M(\alpha)$ in contrast to the corresponding element in the intensity vector of game $M(\beta)$, or, add the restriction $y_j=0$ if the price vector of game $M(\beta)$ contains a 0 in contrast to the price vector of game $M(\alpha)$.

This corresponds to cancelling a row or a column vector of the pay-off matrix. After a finite number of steps the algorithm leads to a solution. At least if $m=n=1$.

The condition (***) excludes the undesirable situation that row or column vectors become zero vectors. The importance of the condition (***) will be shown by the counter example to the constructive proof of O. Morgenstern-G. L. Thompson.

Counter example: In the following example the KMT-condition (***) as well as the modified condition (1.6′) are fulfilled.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 \\ 3 & 2,5 \end{bmatrix} \qquad \mathbb{W} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad \begin{matrix} c = 0 \\ s = 1 \end{matrix}$$

Let $A + c\mathbb{W} = A_1$ and $A + s\mathbb{W} = A_2$ then (1.1–1.5) can be rewritten in the following way:

(1.1′) $$x(B - \alpha A_1) = xM(\alpha) \geqq 0$$

(1.3′) $$x(B - \alpha A_1)y = xM(\alpha)y = 0$$

(1.2′) $$(B - \beta A_2)y = M(\beta)y \leqq 0$$

(1.4′) $$x(B - \beta A_2)y = xM(\beta)y = 0$$

(1.5) $$xBy > 0.$$

The solution to $M(\alpha)$ is: $\alpha = 3$, $y = (1, 0)'$ and $0 \leqq x_1, x_2 \leqq 1$ hence $x = (0, 1)$.

The solution to $M(\beta)$ is: $\beta = 2, 5$, $y = (0, 1)'$, and $x = (0, 1)$.

In accordance with 2 this requires to put $y_1 = 0$, which is equivalent to deleting the first column of the matrices. The new

$$B^{(1)} - \alpha A^{(1)} = M(\alpha)^{(1)} = \begin{bmatrix} 2 - \alpha.0 \\ 2,5 - \alpha.0 \end{bmatrix}$$

The game described above is no longer a zero-sum-game. The new computation is: $xBy = 0$.

The condition (**) is not sufficient to guarantee the existence of a solution.

## 2. J. ŁOS 3-MATRIX-MODEL

From an idea not fully worked out by M. Morishima [31] J. Łos [11] developed an explicit 3-matrix-model. The reasoning was presented as follows:

If $A, B$ are the input and output matrices (technological coefficients: $m \times n$), $1 > 0$ $(m \times 1)$ the vector of the labor input, $c \geqq 0$ the consumption good vector, and $w$ a real number denoting the wage rate paid per unit of labor, then

$$x(A + 1c)$$

describes an input bundle. Considering the problem from the point of view of prices, the cost becomes

$$Ay + w1.$$

The manner in which these vectors are derived is different. Of yet greater importance, the cost vector is not obtained by transformation of the price vector $y$ by a matrix.

The usual way of overcoming these difficulties is by calculating the wage-rate $w$ as the inner product of the consumption good vector $c$ $(1 \times n)$ and prices $y$:

$$w = cy \tag{2.1}$$

Taking this into account, the cost vector is calculated by transforming the price vector by a matrix:

$$Ay + w1 = Ay + 1(cy) = (A + 1c)y.$$

That matrix is the same as the one used to transform the intensity vector $x$ into input bundles.

The v. Neumann Model may now be written in the usual way: $(A + 1c, B)$. Some problems arise: Obviously $cy$ is only meaningful if $cy > 0$.

To assume strict positivity for $c$ and by that ascertaining $cy > 0$ is a very strong assumption; indeed, too strong an assumption to work with. Thus it is possible to find a semipositive vector $y \geqq 0$ which fails to satisfy the above-mentioned necessary condition. Another problem—maybe of even greater importance—occurs if we investigate the equilibrium points of the following KMT-model:

$$xB \leqq \alpha x(A + 1c) \tag{2.2}$$

$$By \leqq \beta(A + 1c)y \tag{2.3}$$

$$x(B - \alpha(A + 1c))y = 0 \tag{2.4}$$

$$x(B - \beta(A + 1c))y = 0 \tag{2.5}$$

$$xBy > 0. \tag{2.6}$$

As $A_i \geqq 0$, $B^j \geqq 0$ (the KMT-condition) allow the existence of subeconomies it may occur that $cy > 0$ can only be fulfilled at the highest equilibrium point. All consumer goods with positive prices at the highest level may become free goods at lower levels and in doing so not fulfill $cy > 0$.

T. Bromek dealt with this difficulty by investigating the problem under the assumption that $cy > 0$, and he obtained extremely interesting results (3, 2). J. Łos approached the problem in the following way:

He accepts the input matrix $A_1 = (A + 1c)$, but defines a different cost matrix $A_2$. Introducing $e = (1, \ldots 1)$, a summation vector $(1 \times n)$ we get:

$$ey = 1.$$

$$wley = w1$$

$$Ay + w1 = Ay + wley = (A + wle)y = A_2.$$

In the most general form J. Łos proved the following theorem:

THEOREM 4: Assuming $A_{1i} \geqq 0$, $A_2 > 0$, $B^j \geqq 0$ then there exist solutions $x \geqq 0$, $y \geqq 0$, $\alpha > 0$, $\beta > 0$ to (2.7–2.11).

(2.7) $$xB \geqq \alpha x A_1$$

(2.8) $$By \leqq \beta A_2 y$$

(2.9) $$x(B - \alpha x A_1)y = 0$$

(2.10) $$x(B - \beta x A_2)y = 0$$

(2.11) $$xBy > 0.$$

PROOF: The outline of the proof given by J. Łos follows the train of thoughts of the v. Neumann proof of the existence of equilibrium in the original model (11). The proof uses the rather massive tool of fixed point theorems.

It is obvious that the model specified above can be expressed as 2 games (2-person-zero-sum games):

(2.12) $$M(\alpha) = (B - \alpha A_1) \qquad v(M(\alpha)) = 0$$

$$x(B - \alpha A_1) = xM(\alpha) \geqq 0$$

(2.13) $$M(\beta) = (B - \beta A_2) \qquad v(M(\beta)) = 0$$

$$(B - \beta A_2)y = M(\beta)y \leqq 0$$

Under condition (~~xxx~~) one can apply Theorem 3.*

REMARK: The assumptions $A_{1i} \geqq 0$, $B^j \geqq 0$ are equivalent to the usual KMT-conditions. To ascertain $cy > 0$, the assumption $A_2 > 0$ is mandatory; only in this way the mapping on $y$ can be adequately defined.

## 3. 4-MATRIX-MODEL

The different input matrices in the MT-model were connected by the assumption of the $(m \times n)$ matrix $W$. The real numbers $(0 \leqq c, s \leqq 1)$ constitute the difference.

In fact, the MT-model constitutes already a 3-matrix-model. J. Łos' 3-matrix-model considering different input matrices with respect to the real or monetary world, was developed without knowledge of the MT-approach. The economic significance of the 3-matrix-model approach is not to be underestimated. The interpretation shows that a solution may exist even if there is an optimal allocation in the real world, and an optimal allocation in the monetary world, which are not based on the same entries in the input-matrix, suggesting that different evaluations may be applied to different aims.

It may be worthwhile to allow for different "evaluations of utility" not only in the input matrices. It appears to be more convenient to consider a real world (technology) in contrast to a socio-political

world (expressed in money). The following question then remains to be answered: Is it possible for equilibrium considerations to incorporate the practically incongrous technological and social structures? Disequilibrium.

We think that it is worthwhile to consider the connection between equilibrium and disequilibrium on the one side and solutions not necessarily efficient on the other hand. We will devote one of the following chapters to efficiency considerations.

The considerations in the preceding paragraphs suggest that a 4-matrix-model is called for both on economic as well as on purely formal grounds.

DESCRIPTION: In a 4-matrix-model we have 4 nonnegative matrices ($A_1$, $A_2$, $B_1$, $B_2$) and as many transformations. (In the original v. Neumann model we have two matrices, but four transformations). As left operators $\cdot A_1$, $\cdot B_1$ transform the intensity vector (probability vector) into input $xA_1$ and output $xB_1$ bundles. As right operators, $A_2\cdot$, $B_2\cdot$ transform the normalized price vector into cost vectors and into return vectors $A_2y$ and $B_2y$, respectively. The equilibrium for the above mentioned model can be presented in the following way:

(3.1) $$x(B_1 - \alpha A_1) \geqq 0$$

(3.2) $$(B_2 - \beta A_2)y \leqq 0$$

(3.3) $$x(B_1 - \alpha A_1)y \equiv 0$$

(3.4) $$x(B_2 - \beta A_2)y = 0$$

(3.5) $$xB_1y > 0; \qquad xB_2y > 0.$$

The above mentioned model (3.1–3.4) can be transformed into two games:

(3.6) $$M(\alpha) = (B_1 - \alpha A_1) \qquad v(M(\alpha)) = 0$$

$$x(B_1 - \alpha A_1) = xM(\alpha) \geqq 0$$

(3.7) $$M(\beta) = (B_2 - \beta A_2) \qquad v(M(\beta)) = 0$$

$$(B_2 - \beta A_2)y = M(\beta)y \leqq 0.$$

LEMMA 5: If there exist solutions $x$, $y$, $\alpha$, $\beta$, to the above stated system of equalities and inequalities (3.1–3.5) then

(3.8) $$\alpha = xB_1y/xA_1y > 0$$

(3.9) $$\beta = xB_2y/xA_2y > 0$$

PROOF: Since $xB_1y > 0$; from $xB_1y = \alpha xA_1y$ it follows that

$$xA_1y > 0 \qquad \text{and} \qquad \alpha > 0.$$

Since $xB_2y > 0$; from $xB_2y = \beta xA_2y$ it follows that

$$xA_2y > 0 \qquad \text{and} \qquad \beta > 0.$$

THEOREM 6: For a model (3.1–3.5) fulfilling the following condition

(xxx) $$A_{1i}^{(j)} \geqq 0, \qquad B^j{}_{1(i)} \geqq 0; \qquad A_{2i}^{(j)} \geqq 0, \qquad B^j{}_{2(i)} \geqq 0$$

there always exist solutions $x \geqq 0, y \geqq 0, \alpha > 0, \beta > 0$; possibly more than one.

PROOF (the main concept is the same as in the MT-Proof) (25):

Part 1: Let $x$ and $\alpha^*$ be solutions to $M(\alpha)$ and $y$ and $\beta^*$ solutions to $M(\beta)$. If the positive components of $x$ correspond to the positive components of the intensity vector of game $M(\beta)$ and the positive components of $y$ correspond to those components of the price vector of game $M(\alpha)$, then $x$, $y$, $\alpha^*$, $\beta^*$ are solutions.

Proof to Part 1: Consider the reasoning applied to Theorem 3 resp. Theorem 3*. It fulfills the game interpretation of the model (3.1–3.5). It must be mentioned that the assumption of part 1 need not be fulfilled.

Part 2: If the assumptions of part 1 are not fulfilled there still exist solutions to the model (3.1–3.5).

Proof to Part 2: If $m = n = 1$ it is obvious that there exists a solution to the model. If not, try to solve both games $M(\alpha)$, $M(\beta)$ as usual v. Neumann models. If there are no solutions fulfilling the assumptions of part 1 then introduce an additional restriction: If there exists an element 0 in the intensity vector $x$ of $M(\alpha)$ which is opposed to the corresponding element in the intensity vector of $M(\beta)$, then add the restriction $x_i = 0$. Or, add the restriction $y_j = 0$, if the price vector of $M(\beta)$ contains a 0 which is opposed to the price vector of $M(\alpha)$. This procedure is equivalent to cancelling a row or a column vector of the pay-off matrix.

If the solution of the reduced game leads to a situation in conformity with the assumption of part 1, then the problem is solved. If not, continue with the procedure of part 2. After a finite number of steps the algorithm leads to a solution, at least if $m = n = 1$. By taking into account all possible ways of introducing restrictions, all possible solutions to (3.1–3.5) can be found. Obviously there need not be a unique solution.

REMARK 1: If instead of condition (~~xxx~~) condition (~~xx~~) is assumed then only subeconomies of $M(\alpha)$ resp. $M(\beta)$ each containing at least one equal positive good and one equal positive process have solutions; they can be derived by using the above mentioned algorithm.

REMARK 2: If there are no subeconomies with regard to 1 then a solution does not exist.

It is worth mentioning that the 4-matrix-model specified above implies two different approaches to the theory of value: Game $M(\alpha)$ may represent the labor value theory, while game $M(\beta)$ is based on the utility concept. An interpretation of the model may not be without interest.

The v. Neumann model can be interpreted as a game in which player 1 is an association of entrepreneurs, player 2 an association of unions or a central union, both operating in a market economy.

In a planned society the maxmin player can be thought of as a ministry of production, the minmax player a ministry of income and/or social affairs.

In the 4-matrix-game we will retain the interpretation as far as the two players are concerned. But these players are now engaged in two different games: Consider $M(\alpha)$ as production game and $M(\beta)$ as a society or ideological game. The solution will be determined by the strategies of the ministry of production in the production game and the strategies of the ministry of social affairs in the ideological game.

## 4. EXAMPLE

The importance of condition (⁂) may be demonstrated by taking a historical economic model where only condition (⁂) is assumed to be operative. This "experiment" may yield some interesting insights and open up new perspectives.

One of the most fascinating cultures in history that we know of is the Maya culture in central America. Being primarily a Stone Age society, the Mayas obtained fantastic results in astronomy, surpassing—possibly—even the famous Egyptians.

Beside the still unsolved puzzle of their undecipherable hieroglyphs, their disappearance in the ninth century A.D. represents a further enigma to scholars interested in their history. So far no one has been able to discover why the Mayas disappeared: No traces of war, civil war, earthquakes, diseases or of exhausted soil could be found.

The following might be a possible explanation: The Maya-society consisted of two groups: the ruling class of priests and the class of corn-growing farmers. The governing class of priests possibly lost their grip on the farmers. The cause of this loss of authority could either be attributed to excess of concentration on astronomy and/or lack of consideration for the farmers need; maybe that the thrill of scientific discoveries was of much higher importance to the priests than the indirect profit to the production process of corn-growing carried on by the farmers. Given this situation a linear homogeneous interpretation may be found. By using the 4-matrix-model, $A_1$, $B_1$ describe the technology, and $A_2$, $B_2$ the evaluation of different tastes in a society, naturally with regards to the ruling class. (The famous remark of Marie Antoinette upon hearing about the lack of bread in Paris: "Why don't they eat cakes?" characterizes the disintegration of both the technological necessities and the values of a society.)*

Technology-matrices:

| | $A_1$ | | $B_1$ | |
|---|---|---|---|---|
| corn growing | 10 | 0 | 20 | 0 |
| temple building | 50 | 0 | 0 | 1 |

Transform this v. Neumann model of an expanding economy into a game

$$M(\alpha) = (B_1 - \alpha A_1)$$

$$xM(\alpha) \geqq 0 \qquad vM(\alpha) = 0$$

*This is characteristic of a situation where technology and the system of values of a society have parted company.

we obtain the solutions: $\alpha=\beta=2$; $x=(1, 0)$, $y$ is arbitrary, $1/101 \leqslant y_1 \leqslant 100/101$; $xB_1y>0$ is fulfilled. The process "temple-building" is inefficient and will not be employed (from a technological point of view)

$$M(\alpha)=\begin{bmatrix} 0 & 0 \\ -100 & 1 \end{bmatrix} \qquad xM(\alpha)=(1, 0)\begin{bmatrix} 0 & 0 \\ -100 & 1 \end{bmatrix} \geqq 0$$

$$xM(\alpha)y=(1, 0)\begin{bmatrix} 0 & 0 \\ -100 & 1 \end{bmatrix}\begin{bmatrix} 1/101 \\ 100/101 \end{bmatrix}$$

Taste matrices:

| | $A_2$ | | $B_2$ | |
|---|---|---|---|---|
| | corn | temple | corn | temple |
| corn growing | 0 | 1 | 20 | 0 |
| temple building | 0 | 1 | 0 | 2 |

Similarly, the v. Neumann model of an expanding economy (tastes) is transformed into a game

$$M(\beta)=(B_2-\beta A_2)$$

$$M(\beta)y \leqq 0 \qquad vM(\beta)=0$$

and the following results: $\alpha=\beta=2$; $x=(0, 1)$, $y$ is arbitrary $10/11 \leqslant y_2 \leqslant 1$ are obtained. $xB_2y>0$ is fulfilled. Obviously corn-planting was not one of the prime occupations of the ruling class of priests.

$$M(\beta)=\begin{bmatrix} 20 & -2 \\ 0 & 0 \end{bmatrix} \qquad M(\beta)y=\begin{bmatrix} 20 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/11 \\ 10/11 \end{bmatrix} \leqq 0$$

$$xM(\beta)y=(0, 1)\begin{bmatrix} 20 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/11 \\ 10/11 \end{bmatrix}=0.$$

It is obvious that the assumptions of part 1 of Theorem 6 are not fulfilled.

Add the restriction on $M(\alpha)$ resp. $M(\beta)$:

$$\begin{aligned} x_1=0 &\Rightarrow x_2=1; \quad y_1=1,\ y_2=0; \quad \alpha=0; \quad xB_1y=0 \\ &\Rightarrow x_2=1; \quad y_1=0;\ y_2=1; \quad \beta=2; \quad xB_2y>0. \end{aligned}$$

There exists no solution satisfying both games at the same time. On the one hand, the technological structure no longer supported the socially accepted utility concept, on the other hand the society did not appreciate (more likely not accept) the given technology.

## 5. PRESENTATION OF v. NEUMANN MODELS AS ONE OR TWO 2-PERSON-ZERO-SUM-GAMES

Reformulating v. Neumann's models with one or two games may yield some insight into underlying assumptions, which in some cases may be additional ones that are absolutely necessary.

(a) Demand was first introduced into the model by J. G. Kemeny, O. Morgenstern and G. L. Thompson [9] (aside from repeated interpretations of the input coefficient) requiring that at each time period the economy should supply a $(1 \times n)$ vector $d$ to an outside consumer, which may be thought of as additional consumption of the employees. The decisive assumption is that the consumption vector $d$ is always a constant fraction of the output. Additional outside demand exhibits the same growth rate as the whole economy. As long as consumer goods have a positive price it is obvious that the opportunity costs of consumption is commensurate with the growth rate.

$$x(B - \alpha A) \geqq \alpha d \tag{5.1}$$

$$(B - \beta A)y \leqq \beta e(dy) \tag{5.2}$$

$$x(B - \alpha A)y = \alpha dy \tag{5.3}$$

$$x(B - \beta A) = \beta dy \tag{5.4}$$

$$xBy > 0 \tag{5.5}$$

By defining $e$ as a $(n \times 1)$ summation vector, with each element of the vector being one, the model can easily be formulated by either one or two games.

(5.6)

$$M(\alpha) = (B - \alpha(A + ed)) \qquad M(\beta) = (B - \beta(A + ed))$$

$$xM(\alpha) \geqq 0 \qquad M(\beta)y \leqq 0$$

$$vM(\alpha) = vM(\beta) = 0 \qquad \alpha = \beta.$$

(b) E. Malinvaud [15] based his analysis on the KMT way of introducing additional (external) consumption. Notwithstanding some criticism of the way the price vector is introduced in KMT, arguing that owing to the existence of positive profits the assumption would not be compatible with a competitive equilibrium.

E. Malinvaud defines a "program over a horizon $h$" as a sequence of vectors (16) $\{a_0, x_0; b_1, a_1, x_1 \ldots; b_t, d_t, a_t, x_t \ldots .; b_{h-1}, d_{h-1}, a_{h-1}, x_{h-1}; b_h\}$ fulfilling

$$a_t \geqq x_t A, \tag{5.7}$$

$$b_{t+1} \leqq x_t B. \tag{5.7'}$$

As a result of the assumption of "free disposal (5.7, 5.7') a program is adequately described by $(\alpha, x, d)$.

A program $(\alpha, x, d)$ is said to be maximal if the expansion factor is the greatest growth factor for all possible programs satisfying

(5.1) $$x(B-\alpha A) \geqq d \qquad x \geqq 0, \qquad \alpha > 0.$$

Associated with a maximal program $(\alpha, x, d)$ is the following model (17):

(5.1) $$x(B-\alpha A) \geqq \alpha d$$

(5.8) $$(B-A)y \leqq \alpha q h y$$

(5.9) $$x(B-A)y = \alpha d y$$

Instead of the summation vector $e$, a weighted nonnegative $(n \times 1)$ vector $q$ is used. $d$ again describes a $(1 \times n)$ consumption vector: $d = xqh$; $x$ is the intensity vector, $h$ a $(1 \times n)$ final demand vector. It is easy to see that the above mentioned (maximal) model can be transformed into one game:

(5.10) $$M(\alpha) = (B-\alpha(A+qh))$$

$$xM(\alpha) \geqq 0 \qquad M(\alpha)y \leqq 0 \qquad vM(\alpha) = 0.$$

In contrast to the above mentioned concept, the consumption vector $e$ in E. Malinvaud's efficiency considerations is not connected with the intensity vector.

A program $(\alpha^*, x^*, d^*)$ is defined as "efficient over a horizon $h$" if there is no program whatsoever over the same horizon $h$ so that (18)

(5.11) $$d_1 \geqq \alpha^{*t} d^*,$$

$$a_0 \leqq a^* \qquad \text{and} \qquad b_h \geqq \alpha^{*t} b^* \qquad t = 1, 2, \ldots h-1$$

with at least one strict inequality.

In connection with this concept of efficiency E. Malinvaud defines the following model (19):

(5.12) $$x(B-\alpha A) = \alpha d$$

(5.13) $$(B-\beta A)y \leqq 0$$

(5.14) $$x(B-\beta A)y = 0$$

It can easily be recognized that the model can be reduced to one game only if $d$, a $(1 \times n)$ consumption vector, is zero. In all other cases it can be interpreted by two games only:

(5.15) $$M(\alpha) = (B-\alpha(A+ed)) \qquad M(\beta) = (B-\beta A)$$

(5.16) $$xM(\alpha) \geqq 0 \qquad M(\beta)y \leqq 0 \qquad vM(\alpha) = vM(\beta) = 0.$$

(c) M. Morishima's [28] generalized v. Neumann Model (Marx-v. Neumann, Walras-v. Neumann) using a two-class-society concept by introducing wages and profits can be interpreted as one game, if and only if the income of the entrepreneurs, which is not spent for consumption, (thus the saved amount) is disposed of outside the analysed economy. If the amounts saved are spent within the same economy it can only be transformed into two games.

The original M. Morishima model deals de facto only with consumption, not with income. Since the income of employees, who spend all their money on consumption, and that of employers, who spend a portion of their income on consumption are separated, the result—under certain conditions—is a mere reduction of the growth rate in comparison with the interest factor.

(5.17) $$xB \geqq \alpha(xa + x\omega Lg(y)) + \max(0, \beta - 1)(xA_y + x\omega L)f(y)$$

(5.18) $$By \leqq \beta(Ay + \omega L)$$

(5.19) $$xBy = (\alpha + \max(0, \beta - 1)c)(xAy + x\omega L)$$

(5.20) $$xBy = \beta x(Ay + \omega L)$$

(5.21) $$xBy > 0$$

$\omega$ is the real wage rate, $L$ is a $(m \times 1)$ strictly positive vector of labor input, $f(y)$ and $g(y)$ are $(1 \times n)$ vectors so that $f(y)y = c$, $g(y)y = 1$ characterizing the Engel-elasticity of capitalists, respectively the employees' consumption of goods. Further it is assumed that $f(y)$ and $g(y)$ are nonnegative. All $x$ and $y$ are in the standard simplices.

(5.17) implies that the goods are consumed either in the process of production or through employees' or capitalists' consumption.

(5.18) implies that no process must yield excessive profits; the unit cost of production includes the interest on constant capital as well as on variable capital (labor).

The model can be transformed into a game, defining ($s$ = savings ratio)

(5.22) $$M(y, \alpha) = A + \omega Lg(y) + \max\left(0, \frac{\alpha - 1}{\alpha s}\right)(Ay + \omega L)f(y)$$

$$x(B - \alpha M(y, \alpha)) \geqq 0$$

$$(B - \alpha M(y, \alpha))y \leqq 0$$

(5.21) $$xBy > 0$$

Dividing $B - \alpha M(y, \alpha)$ by $1 + \alpha$ and denoting $\frac{\alpha}{1 + \alpha} = \lambda$ the above expression becomes

$$((1 - \lambda)B - \lambda N(y, \lambda))y \leqq 0$$

$$x((1-\lambda)B-\lambda N(y,\lambda)) \geqq 0$$

$$xBy > 0$$

(5.23) $$N(y,\lambda) = A + \omega Lg(y) + \max\left(0, \frac{\Omega\lambda - 1}{s\lambda}\right)(Ay + \omega L)f(y)$$

Furthermore M. Morishima concludes, that $N(y, \lambda)$ is continuous for all $(y, \lambda)$ in

$$S = \left\{(y,\lambda) | y_j \geqq 0, \sum_j^n y_j = 1 \quad \text{and} \quad 0 \leqq \lambda \leqq 1\right\}.$$

The above is not derived from the definitions. The difficulties arise from the demand vectors.

(5.24) $$d(W, y) = (W)g(y) \qquad W = dy \ . \ . \ . \ \text{income}$$

$g$ is continuous on $S^n : R^n - \{\theta\} \to R^n$, a real valued function. It is not possible however to deduce from the assumption of the model that not all consumer goods take on a price $y_j = 0$, which means the consumer goods become free goods. It is not specified that the mapping $g \neq 0$. Thus it may be impossible to define the income as the inner product of the demand vector and the price vector.

This obviously implies that it need not to be true that "once the real wage rate is given, all prices, the rate of interest and the rate of growth are determined" (29).

There are two alternative ways of avoiding these difficulties. Concentrate the analysis on two cases:

(5.25) $$dy > 0, \qquad W > 0; \qquad W = 0.$$

The latter case reduces the model to the classical KMT model. An alternative approach is the assumption of a consumer goods vector $(1 \times n)\, d > 0$.

This shows why J. Łos [12] in his 3-matrix-model had to assume that $A_2 > 0$ (cost matrix strictly positive).

The first case leads to the question put forward and solved by T. Bromek [3].

THEOREM 9: As the models (5.1–5.5), (5.1, 5.8, 5.9), (5.12–5.14), (5.17–5.21) are transferable into (5.6), (5.6), (5.10), (5.15–5.16), and (5.23) if (5.25) is satisfied, there exist solutions $x \geqq 0$, $y \geqq 0$, $\alpha > 0$, $\beta > 0$ to these models.

PROOF: Follows the proof of Theorem 6.

## 6. EFFICIENCY, TWO-PERSON-ZERO-SUM-GAMES AND v. NEUMANN MODELS

J. G. Kemeny, O. Morgenstern, G. L. Thompson [7], J. G. Kemeny [6], G. L. Thompson [32] have shown that the v. Neumann model can be interpreted as one 2-person-zero-sum-game. O. Morgenstern, G. L. Thompson [22] showed, that it is possible to interprete some extensions of v. Neumann models as two 2-person-zero-sum-games. Even though game theory was originally considered to be a purely mathematical device, as in KMT, it can also be thought of as a decisive guide to interpretation.

This new tool proved to be very fruitful in dealing with v. Neumann models. To gain deeper insight and understanding of the underlying models, it may be rewarding to look for some other tool—even if it will not prove to be as useful as the one mentioned above.

By using E. C. Koopmans [10] studies of efficiency in linear economic models, generalized by G. Debreu [4] and extended to models with infinite horizon by E. Malinvaud [13, 21] as basis, v. Neumann models will be dealt with from the point of view of efficiency. Instead of dealing primarily with the reproductive character of different periods (scalar multiplication with a factor) the time aspect of the model will now take priority. For this reason a subscript denoting time will frequently be used. Consider the inequalities and equalities of the v. Neumann model ($W=0$ and $x_t=x$, $y_{t+1}=y$ as in the model of chapter 1):

(1.1) $$xB \geqq \alpha xA$$

is equivalent to the following conditions:

(6.1) $$x_t=\alpha x_{t-1} \qquad x_t \geqq 0 \qquad t=1, \ldots, h$$

(6.2) $$x_tA \leqq a_t$$

(6.3) $$a_{t+1}=x_tB$$

The growth of the economy will result from the multiplication of the intensity vector (6.1) with a scalar. Starting a program at time $t$ with a $(1 \times n)$ stock vector $a_t$ (6.2) follows as consequence of the assumed property: "free disposal." Assuming that the output of one period $t$ will be completely used by the input in the next period $t+1$, leads to (6.3).

(1.2) $$By \leqq \beta Ay$$

can be reduced to the following assumptions:

(6.4) $$\beta y_{t+1}=y_t$$

(6.5) $$By_{t+1} \leqq Ay_t$$

(6.4) implies the equivalence of the not discounted price vectors and (6.5) shows, that the money value of the output is limited by the interest (discounting factor) on the inputs.

(1.3) $$x(B-\alpha A)y=0$$

the "rule of free goods" can be reduced to the following assumptions: (6.1) and

(6.6) $$a_{t+1} \leqq x_tB$$

(6.7) $$a_{t+1}y_{t+1}=x_tBy_{t+1}$$

The condition that inefficient processes will have a zero-intensity

(1.4) $$x(B-\beta A)y=0$$

is equivalent to (6.4) and

(6.8) $$x_tBy_{t+1}-x_tAy_t=0.$$

Moreover there cannot exist an $x_t^*$, $t=1. \ . \ . \ ., h$ fulfilling

(6.9) $$x_t^* \leqq x_t$$

(6.10) $$x_t^*A \geqq x_tA$$

and for all goods produced in the last period the price should be

(6.11) $$y_{t+1} > 0.$$

As this assumption is rather inconvenient, let us replace it by the less strict condition

(6.11) $$y_{t+1} \geqslant 0.$$

A sequence of vectors $y_t$ satisfying (6.2, 6.5, 6.6, 6.7) is not necessarily sufficient for a program $x$ to be efficient. But since it can be shown that the prices of elements of $x_tB$ are zero only if they can be increased without sacrificing some other goods the program can be called quasi-efficient. We will, however, ordinarily use the term efficient. The problem of finding $(x, y, \alpha=\beta)$ to satisfy the model (1.1–1.5) as analyzed by v. Neumann can be equally formulated as the problem of finding an efficient program expanding at a proportional rate. It can be shown that $\alpha$ and $\beta$ is the saddle point of $\frac{xBy}{xAy}$

(6.12) $$\alpha=\beta=\frac{x^*By^*}{x^*Ay^*}=\max_{x\geqslant 0}\min_{y\geqslant 0}\frac{xBy}{xAy}.$$

LEMMA 10: (As v. Neumann already showed) $\alpha$ is the maximum expansion factor for the given technology while $\beta$ is the minimum interest factor satisfying the profitless condition.

PROOF: By referring to the H. W. Kuhn-A. W. Tucker theory that assertion can easily be proved

$$\max \alpha$$

subject to

$$x(B-\alpha A) \geqq 0 \qquad x \geqslant 0.$$

Solve the problem by using the Lagrangean method

(6.13) $$f(\alpha, x, y) = \alpha + x(B - \alpha A)y.$$

The necessary conditions for $(\alpha^*, x^*, y^*)$ to be a saddle point can be seen when observing the first derivatives

(6.14) $$f_\alpha = 1 - x^* A y^* = 0$$

(6.15) $$f_x = (B - \alpha^* A) y^* \leqq 0$$

(6.16) $$f_y = x^*(B - \alpha^* A) y^* \geqq 0$$

(6.17) $$x f_x = x^*(B - \alpha^* A) y^* = 0$$

(6.18) $$f_y y = x^*(B - \alpha^* A) y^* = 0 \qquad x \geqq 0 \qquad y \geqq 0$$

The (in)equalities (6.15–6.18) correspond to the v. Neumann (in)equalities provided that only $\alpha^*$ occurs. That does not matter, since from (1.3, 1.4) it follows that $\alpha^* = \beta^*$.

The additional condition (6.14) is of absolutely no importance for the assumptions of linear homogeneity (dividing the price vector of any solution by the value of inputs does not in the least alter the solution). A similar proof may be given for min $\beta$. Thus it is verified that the v. Neumann conditions are the necessary conditions for max $\alpha$.

## 7. REMARKS ON SOME MODELS (E. MALINVAUD AND MT)–BASED ON THE EFFICIENCY CRITERION

(a) E. Malinvaud: (maximal and efficient programs)

(aa) Consumption rule: As mentioned before in chapter 5b, the definition connecting $(1 \times n)$ consumption vector $d$ with the intensity vector is decisive for the formulation of the properties of the maximal program. The consumption vector therefore expands at the same rate as the whole economy.

The connection between the consumption vector and the intensity vector will not be dealt with. Furthermore if $b_{t+1} = x_t B$ $(t = 0, 1, \ldots h)$ and $a_t$ is equal to the stock vector available as input (time $t$) then

$$b_t = a_t + d_t \qquad (t = 1, \ldots h).$$

Treating the efficiency model as a v. Neumann model corresponds to the maximization under the efficiency criterion, as shown in the previous section. The only exception is (6.3). Instead consumption is taken into account.

(7.1) $$a_{t+1} = x_t B - d_{t+1}.$$

The previous discussion of efficiency did not consider consumption while E. Malinvaud allows for it. The efficiency considerations will influence the concept of consumption. The consumption definition in the maximality consideration (5.7–5.9) implies that overproduced goods (hence free goods) will not

be demanded. This somehow is opposed to the more or less conventional economic assumption of no saturation point.

Considering efficient programs from another angle reveals the following: If there are overproduced goods, efficiency is not possible (as it is implied by the efficiency condition (5.12)).

Even if there are no overproduced goods a maximal program according to E. Malinvaud is not necessarily efficient (counter example in (20)). It may be possible to find another program with the same expansion factor, allowing however for higher consumption. Maximality and efficiency models may be distinguished by some further criterion.

(ab) Profit rule: One of the considerations deals with an analysis of profit. As mentioned before E. Malinvaud questions the compatibility of profit and the concept of a competitive economy (contrary to the efficiency model) in the maximality model. A competitive economy requires a profitless system. E. Malinvaud's doubts are based on $(B-\beta A)y \leqq qhy$ (5.8). Implicitly the price system $y$ does not seem to be meaningful. This implies that

(5.8) $$(B-\beta A)y \leqq \beta qhy$$

(5.1) $$x(B-\alpha A) \geqq \alpha d \qquad d=xqh$$

is only mathematically, but not economically, equivalent to

$$(B-\beta(A+qhy))y \leqq 0$$

$$x(B-\alpha(A+qh)) \geqq 0.$$

The first representation would offer a meaningless price system, the second would not. Allowing for additional consumption $(A+qh)$ implies a different technology.

The additional consumption of the employees, defined by $d$, is present in a closed model. This requires that income corresponds to spending. It follows that both spending and cost of the employees must be equal.

Therefore it becomes evident that the condition of a profitless economy is fulfilled since the amount paid above the subsistence level (additional consumption) enter the profit definition as a negative component.

It is not possible to reject the program on account of the profit rule.

Obviously this does not imply a rejection of $(B-\beta A)y \leqq 0$ (5.13) in the efficiency condition. This condition is an acceptable assumption. It should be noted that the assumed economic model may be different from the one on which the maximality conditions are based. A reasonable explanation to (5.12–5.14) seems to be that the capitalists spend a portion of their profit on consumption.

(ac) Expansion and the interest factor: Another difference between the two programs can be seen through the expansion and interest factor. The expansion factor is optimal for both models (5.1, 5.8, 5.9) (5.12–5.14) if they are both maximal and efficient, and fulfill (20):

(7.2) $$q=Ay, \qquad h=\frac{d}{xAy}.$$

Generally, the value of the interest factor differs in the models (except $d=0$) (5.1, 5.8, 5.9) (5.12–5.14). In (5.1, 5.8, 5.9) interest and expansion factor are equal, while in (5.12–5.14) the following relation exists:

$$\beta-\alpha=\alpha\frac{dy}{xAy}. \tag{7.3}$$

This difference may be caused by models defined in different ways. As mentioned above, the capitalists spend a portion of their profit on consumption. As M. Morishima [30], N. Kaldor and J. Robinson (in the classical tradition) proved that the interest rate will surpass the expansion rate when capitalists consume part of their income.

(ad) Equality of assumption: If the analysis is correct it may be possible to show that the difference between a maximal and an efficient program is not the result of the above criteria, but rather of the different underlying models. The following uniform model is constructed:

Employees will spend their entire income on consumption and employers will not spend any of their profit.

Our reasoning above makes it necessary to attribute to employees an income even in the efficiency model so that there is something to be spent on consumption.

For this reason a $(m\times 1)$ labor vector $1>0$ is introduced, $1_i$ being the necessary input of labor in the $i$th process at unit intensity. $m$ denotes a $(m\times 1)$ vector. The elements show the maximal amount of labor which can be used in the process.

The following restriction can thus be added to E. Malinvaud's approach (e: summation vector $(1\times m)$):

$$x_t 1 \leqq em_t \qquad 1 \leqq m \tag{7.4}$$

Furthermore it is assumed that the labor force expands at the same rate as the model.

In the model $\omega$ is a surplus wage-rate if the so-called subsistence income is already included in the technological coefficients of the input matrix $A$. An instructive interpretation can be given to $\omega=0$: excess supply of labor force.

Since by assumption income of employees is completely spent, the following equation holds true:

$$d_{t+1}y_{t+1}=em_t\omega_t. \tag{7.5}$$

To avoid the difficulties encountered in dealing with M. Morishima (in chapter 5c) the following assumptions are added:

To guarantee a linear mapping $R^n-\{\theta\}\to R$, it is necessary to assume either $d_{t+1}$ or $y_{t+1}$ strictly positive or to make one of the following additional assumptions:

$$y_{t+1}>0, \qquad d_{t+1}\geqq 0 \tag{7.6}$$

$$d_{t+1}>0, \qquad y_{t+1}\geqq 0 \tag{7.7}$$

(7.8) $$d_{t+1}y_{t+1} > 0.$$

Since $y_{t+1}$ is semipositive, (7.7) automatically fulfills (7.8): $\omega > 0$.
Restate the assumptions concerning the model

(7.1) $$x_t B = a_{t+1} + d_{t+1}$$

(5.7) $$x_t A \leqq a_t \qquad t = 0, 1, \ldots h-1$$

(7.4) $$x_t 1 \leqq e m_t \qquad x_t \geqq 0, \qquad d_t \geqq 0.$$

In accordance with this modification the assumptions concerning efficiency can be formulated in the following manner:

THEOREM 11: If there exists an efficient $h$-program $(\alpha, x, d)$ (for any $h$) then there exist a vector $y \geqq 0$, and numbers $\omega \geqq 0$, $\beta > 0$ fulfilling:

(5.7) $$x(B - \alpha A) = \alpha d$$

(7.9) $$(B - \beta A)y - l\omega \leqq 0$$

(7.10) $$x((B - A)y - l\omega) = 0$$

(7.4) $$x1 \leqq em$$

(7.11) $$(x1 - em)\omega = 0.$$

PROOF: The proof is equivalent to the one given for the original model of E. Malinvaud.

Again the profitless condition is fulfilled: income of employees enters the profit definition as a negative component.

LEMMA 12: If $\omega > 0$ and there exists a solution to (7.1, 5.7, 7.4) taking into account (7.5), then $\alpha = \beta$.

PROOF: Subtract Equation (7.11) from Equation (7.10)

$$xBy - \beta xAy - em\omega = 0.$$

Postmultiply Equation (5.7) with the price vector $y$

$$xBy - \alpha xAy - \alpha dy = 0.$$

Subtract these two equations:

(7.12) $$\beta - \alpha = \frac{\alpha dy - em\omega}{xAy}.$$

If (7.5) is fulfilled, then it is obvious that $\beta=\alpha$. This does not take into consideration results in the expression already deduced by E. Malinvaud when dealing with the efficient program.

(b) O. Morgenstern-G. L. Thompson. By analyzing E. Malinvaud's model it can be shown that efficiency and maximality considerations are equivalent, when keeping the underlying model of the economy constant.

It is obvious that the MT model is not in agreement with the efficiency criteria. If assumption (1.1) is made, the following assumptions (1.2–1.4) contradict the efficiency criteria. What then constitutes the difference between a model in the original form and one that is also efficient?

In defining $A+c\mathbb{W}=A^*$, the efficiency condition corresponds to the v. Neumann (in)equalities if $A^*$ is substituted for $A$. The consequence is obvious: In (1.2) and (1.4) it would be sufficient to substitute the propensity to save $s$ by the propensity to consume $c$. If $c=s$ the model is efficient.

Considering (1.2) $s\mathbb{W}$ is a cost matrix: It seems reasonable and economically plausible to interpret $s\mathbb{W}$, in the stock model of v. Neumann (where the equality of saving and investment in accordance with the classical theory should be fulfilled) as an additional stock of goods which may re-enter the production process. In following this reasoning max $\alpha$ can be obtained

$$\text{s.t. } x(B-\alpha(A+(1-s)\mathbb{W}))\geqq 0. \tag{7.13}$$

$x\mathbb{W}$ are the goods available to the economic agents, $sx\mathbb{W}$ is that portion of the goods which will re-enter the production processes. The net amount of goods available to the economic agents is

$$x(1-s)\mathbb{W}.$$

Considering the above mentioned term, it is plausible that $c+s=1$ need not be fulfilled.

The efficiency criteria should have the following form

$$(B-\beta(A+(1-s)\mathbb{W}))y\leqq 0 \tag{7.14}$$

$$x(B-\alpha(A+(1+s)\mathbb{W}))y=0 \tag{7.15}$$

$$x(B-\beta(A+(1-s)\mathbb{W}))y=0 \tag{7.16}$$

$$xBy>0. \tag{1.5}$$

The interpretation of the last two equations (7.15, 7.16) requires a permanent elimination of consumption in favour of saving. The result will be an increase in the expansion factor. It is a form of compulsory saving.

## 8. CONCEPTS OF CONSUMPTION

In the first attempt to classify (5) the introduction of consumption in v. Neumann models two concepts are distinguished: The "consumption-cost" and the "pure" consumption concept. In the "consumption-cost" concept consumption is treated like other costs of production. The KMT-model (outside demand) and the original v. Neumann model represent this category.

In the "pure" consumption concept consumption goods are produced, but they are not part of the input vector. E. Malinvaud's and M. Morishima's model are to be considered as cases in point. A further attempt introduced a new concept—the separate treatment of input and output matrices—in addition to the "consumption-cost" and the "pure" consumption concept (1).

In view of the above a more natural and appropriate way of classification is presented: As was shown the v. Neumann model corresponds to a maximization under constraints (efficiency criterion).

The taking into account of inefficiency, i.e., of events and behaviour outside the defined rules of the games, which takes the model closer to reality, opens up surprising and at the same time promising possibilities. This is especially true in models of the rigidity and high formalization like the v. Neumann model. Let us therefore distinguish 2 attempts to deal with consumption in v. Neumann models: (a) preserving efficiency (b) inefficiency.

ad (a) It is obvious that a model which can be transformed into one game must be efficient. As mentioned above the original M. Morishima model is transformable into one game. (Our additional assumption only dealt with the problem of guaranteeing a solution.)

As already O. Morgenstern-G. L. Thompson showed in a counter example, if entrepreneurs reinvest their savings in the economy—then we may have inefficient solutions. Furthermore, we showed that in an unchanged model efficiency and maximality (E. Malinvaud) are identical.

It is easy to see—as was also shown above—that the original Malinvaud efficiency model can only be reduced to two games. It is inefficient within the theoretical framework of this paper.

ad (b) It is obvious that the MT model, the J. Łos 3-matrix-model and the 4-matrix-model are generally inefficient. Only in extreme cases efficient results will be found.

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# BAYESIAN ESTIMATION: A SENSITIVITY ANALYSIS*

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**ABSTRACT**

The robustness of the assigned prior distribution in a Bayesian estimation problem is examined. A Bayesian analysis for a stochastic intensity parameter of a Poisson distribution is summarized in which the natural conjugate is assigned as the prior distribution of the random parameter. The sensitivity analysis is carried out by assuming the existence of a true prior which is different in form from that of the assigned prior distribution. By using mean-squared error as a measure of performance, the ensuing Bayes decision function is compared to the corresponding minimum variance unbiased estimator. Results indicate that the Bayes estimator is largely robust to deviations from the assigned prior and remains squared-error superior to the MVU type within a broad region.

## 1. INTRODUCTION

Classical estimation procedures yielding maximum likelihood (ML) or minimum variance unbiased (MVU) estimators have been widely used to analyze lifetest data and assess component or system reliabilities. Recently, however, there has been an increased emphasis for the application of Bayesian techniques to estimate reliability and related parameters. See, for example Crelin [1], Drake [2], and Evans [5]. The purpose of this paper is to examine what appears to be a legitimate concern in the application of Bayes' Theorem: the assignment of the prior distribution.

One cannot refute the fact that in a truly Bayesian situation in which the assigned prior is indeed the true prior distribution, the ensuing Bayes decision function is best among all other estimators of a parameter in question. However, since in many problems the actual state of nature is never observed, often one must evaluate and quantify whatever knowledge is available to arrive at a plausible form for the prior distribution. Thus, the assignment of the prior distribution is hardly an exact process and usually reduces to selecting the natural conjugate as the prior for the purpose of expediting an analytical solution. This occurrence has led many, Easterling [3] for one, to cast Bayesian methods as purely academic exercises and to declare the classical techniques as the only legitimate means of analyzing data. However, before Bayesian techniques are summarily laid to rest, it should be determined if the "disease" is terminal.

Suppose there is a true prior distribution depicting the behavior of a stochastic parameter whose form remains unknown. What if the assigned conjugate prior does not accurately gauge the true prior? To what extent then will such an occurrence affect the Bayes decision function? Unfortunately, little is known about the robustness of a Bayes decision function to changes in the conjugate prior. There is, however, a study by Edwards [4] which sheds some light on the subject. In essence, Edwards concludes

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that under certain conditions the prior is dominated by the likelihood function and, thus, an accurate assessment of the true prior may not be necessary.

Since recent discussion such as the one found in [6] has centered on a superficial comparison between Bayesian and classical techniques, it would seem appropriate to attempt a more substantive comparison here between a Bayes decision function and the corresponding MVU estimator in which the assigned prior which gives rise to the Bayes result is not the true prior distribution. The rather straight-forward problem of estimating the intensity parameter of a Poisson distribution is considered. The task is carried out by evaluating and comparing the mean squared error (MSE) of the classical estimator and the Bayes risk of the Bayes estimator to determine the sensitivity of the latter to deviations from the assigned prior distribution.

It should be noted that the Poisson distribution is being used only as a test case here. It follows, therefore, that the results of this study apply solely to this probability law and any temptation to extrapolate from these results without further analysis should be resisted.

## 2. THE POISSON DISTRIBUTION

Consider the Poisson distribution with probability mass function

$$p(x;\, \theta t_s) = (\theta t_s)^x \exp\ (-\theta t_s)/x! \qquad x = 0, 1, 2, \ldots, \tag{1}$$

where $x$ is the number of failures occurring within a time span $t_s$. Of interest is the estimation of the intensity parameter $\theta$.

Clearly, Equation (1) is the likelihood function from which one may easily determine the MVU estimator of $\theta$ to be

$$\hat{\theta} = x/t_s \tag{2}$$

while its mean squared error is

$$\mathrm{MSE}(\hat{\theta}) \equiv E\{(\hat{\theta}-\theta)^2\} = \theta/t_s. \tag{3}$$

Suppose the intensity parameter is a random variable. For a Bayes estimate of the realization $\theta$, assume the natural conjugate as the prior distribution of $\Theta$ which is the gamma with probability density function (pdf)

$$g(\theta) = \frac{\alpha^{\varphi}\theta^{\varphi-1} \exp\ (-\alpha\theta)}{\Gamma(\varphi)} \qquad 0 < \theta < \infty;\ \alpha, \varphi > 0. \tag{4}$$

By invoking Bayes' Theorem, the posterior pdf of $\Theta$ conditional on $x$ is

$$h(\theta|x) = \frac{(t_s+\alpha)^{x+\varphi}\theta^{x+\varphi-1} \exp\{-\theta(t_s+\alpha)\}}{\Gamma(x+\varphi)} \tag{5}$$

Let $\delta(x)$ be a decision rule on $\theta$ and let $L[\theta, \delta(x)]$ be a nonnegative function representing the

loss incurred in choosing $\delta(x)$ as the decision function for $\theta$. It follows, therefore, that if the loss is the squared error function

$$L[\theta, \delta(x)] = [\delta(x) - \theta]^2,$$

the Bayes decision rule is the posterior mean

(6) $$\delta(x) \equiv E(\Theta|x) = \frac{x+\varphi}{t_s+\alpha}.$$

It is well-known that the conditional expectation of the loss for any $\delta(x)$ is the risk, $R[\theta, \delta(x)]$. Since, the conditional probability mass function given by (1) is Poisson and the decision rule is as given by (6), the risk is expressed as

$$R[\theta, \delta(x)] = \sum_{x=0}^{\infty} \left[\frac{x+\varphi}{t_s+\alpha} - \theta\right]^2 \frac{(\theta t_s)^x \exp(-\theta t_s)}{x!},$$

which after some algebraic manipulation reduces to

(7) $$R[\theta, \delta(x)] = \frac{\alpha^2\theta^2 + \theta(t_s - 2\alpha\varphi) + \varphi^2}{(t_s+\alpha)^2}.$$

Moreover, since a Bayes estimator minimizes the expected (Bayes) risk with respect to the assigned prior distribution, $G(\theta)$, the Bayes risk relative to the gamma prior is given by

$$R[G(\theta), \delta(x)] \equiv E\{R[\theta, \delta(x)]\} = \int_0^{\infty} \left[\frac{\alpha^2\theta^2 + \theta(t_s - 2\alpha\varphi) + \varphi^2}{(t_s+\alpha)^2}\right] \frac{\alpha^{\varphi}\theta^{\varphi-1}\exp(-\alpha\theta)}{\Gamma(\varphi)}\, d\theta,$$

which, after integration and simplification, reduces to

(8) $$R[G(\theta), \delta(x)] = \frac{\varphi}{\alpha(t_s+\alpha)}.$$

For the MVU estimator, its MSE averaged over the conjugate prior is

(9) $$\mathrm{MSE}(\hat{\theta}) = \varphi/\alpha t_s.$$

Therefore, the ratio of the Bayes risk to the MSE of the MVU estimator is

(10) $$\frac{R[G(\theta), \delta(x)]}{\mathrm{MSE}(\hat{\theta})} = \frac{1}{1+\frac{\alpha}{t_s}}.$$

Since for squared error loss a Bayes estimator minimizes the MSE with respect to the assigned prior distribution, Equation (10) never exceeds unity. This indicates clearly the known comparative MSE behavior of these estimators in that the Bayes estimator has a substantial squared error advantage as long as the assumptive structure remains valid.

## 3. ROBUSTNESS

As indicated earlier, the conjugate prior has been selected merely for mathematical convenience. If indeed it is the true prior distribution of $\Theta$, Equation (6) is the Bayes decision function while (10) provides the comparative MSE behavior between (2) and (6). More likely, however, the conjugate prior may provide no more than an approximation to the true state of nature.

To evaluate the robustness of the Bayes result to the conjugate prior, assume that the true prior distribution of $\Theta$ is uniform with pdf

$$u(\theta) = 1/\beta, \qquad 0 < \theta < \beta, \tag{11}$$

when the assigned prior is as given by (4). Of course this particular choice is quite arbitrary and is made only to facilitate the investigation.

Equation (7) is the risk derived from the Bayes decision function (6) which is a consequence of the conjugate prior (4). However, since the true prior is different from (4), the change in the risk will be a function of the first two moments of the true prior distribution as depicted by (7). Hence, for (11), $E(\Theta) = \beta/2$, $E(\Theta^2) = \beta^2/3$ and substitution in (7) yields

$$R[u(\theta), \delta(x)] = \frac{2\beta^2\alpha^2 + 3\beta(t_s - 2\alpha\varphi) + 6\varphi^2}{6(t_s + \alpha)^2}. \tag{12}$$

Clearly, Equation (12) is the risk of the decision rule as a function of the parameters of both the assigned prior (4) and the true prior (11). Equivalently, the MSE of the MVU estimator averaged over the uniform distribution is

$$\mathrm{MSE}(\hat{\theta}) = \beta/2t_s.$$

By following the same argument, it is easy to evaluate risks of $\delta(x)$ for some additional probability laws representing true priors of $\Theta$. These are given in Table I.

TABLE I

| True prior of $\Theta$ | Pdf | $R[\bullet, \delta(x)]$ | MSE($\hat{\theta}$) |
|---|---|---|---|
| Uniform | $1/\beta$ | $\frac{2\beta^2\alpha^2+3\beta(t_s-2\alpha\varphi)+6\varphi^2}{6(t_s+\alpha)^2}$ | $\beta/2t_s$ |
| Rayleigh | $\frac{\theta \exp(-\theta^2/2\gamma^2)}{\gamma^2}$ | $\frac{4\gamma^2\alpha^2+\sqrt{2\pi}\gamma(t_s-2\alpha\varphi)+2\varphi^2}{2(t_s+\alpha)^2}$ | $\sqrt{2\pi}\gamma/2t_s$ |
| Normal | $\frac{\exp\left\{-\frac{1}{2\sigma^2}(\theta-\mu)^2\right\}}{\sqrt{2\pi}\sigma}$ | $\frac{(\mu^2+\sigma^2)\alpha^2+\mu(t_s-2\alpha\varphi)+\varphi^2}{(t_s+\alpha)^2}$ | $\mu/t_s$ |
| Extreme value | $\frac{1}{\tau}\exp\left\{-\frac{1}{\tau}(\theta-\upsilon)-\exp\left[-\frac{1}{\tau}(\theta-\upsilon)\right]\right\}$ | $\frac{(1.645\tau^2+A^2)\alpha^2+A(t_s-2\alpha\varphi)+\varphi^2}{(t_s+\alpha)^2}$ $A=\upsilon+0.577\tau$ | $A/t_s$ |

● Prior of $\Theta$

## 4. NUMERICAL COMPARISONS

The ratio of the risk to the mean squared error is formed for each row of Table I and is plotted for selected values of the indicated parameters. Two values for each of the assigned prior parameters $\alpha$ and $\varphi$ are selected. These along with the resulting means and variances of the gamma prior distribution are given in Table II.

Table II

| $\alpha$ \ $\varphi$ | 1.5 | 3 |
|---|---|---|
| 0.3 | mean = 5<br>var = 16.67 | mean = 10<br>var = 33.33 |
| 1 | mean = 1.5<br>var = 1.5 | mean = 3<br>var = 3 |

Figures 1–4 depict the behavior of the ratio as a function of: (1) the scale parameters $\beta$ and $\gamma$ of the uniform and Rayleigh distribution (Figures 1 and 2), and (2) the location parameters $\mu$ and $\upsilon$ of the normal and extreme value distributions (Figures 3 and 4, respectively). In the latter two cases, the values of the scale parameters $\sigma$ and $\tau$ are assigned to be one-third the values of the location parameters $u$ and $\upsilon$.

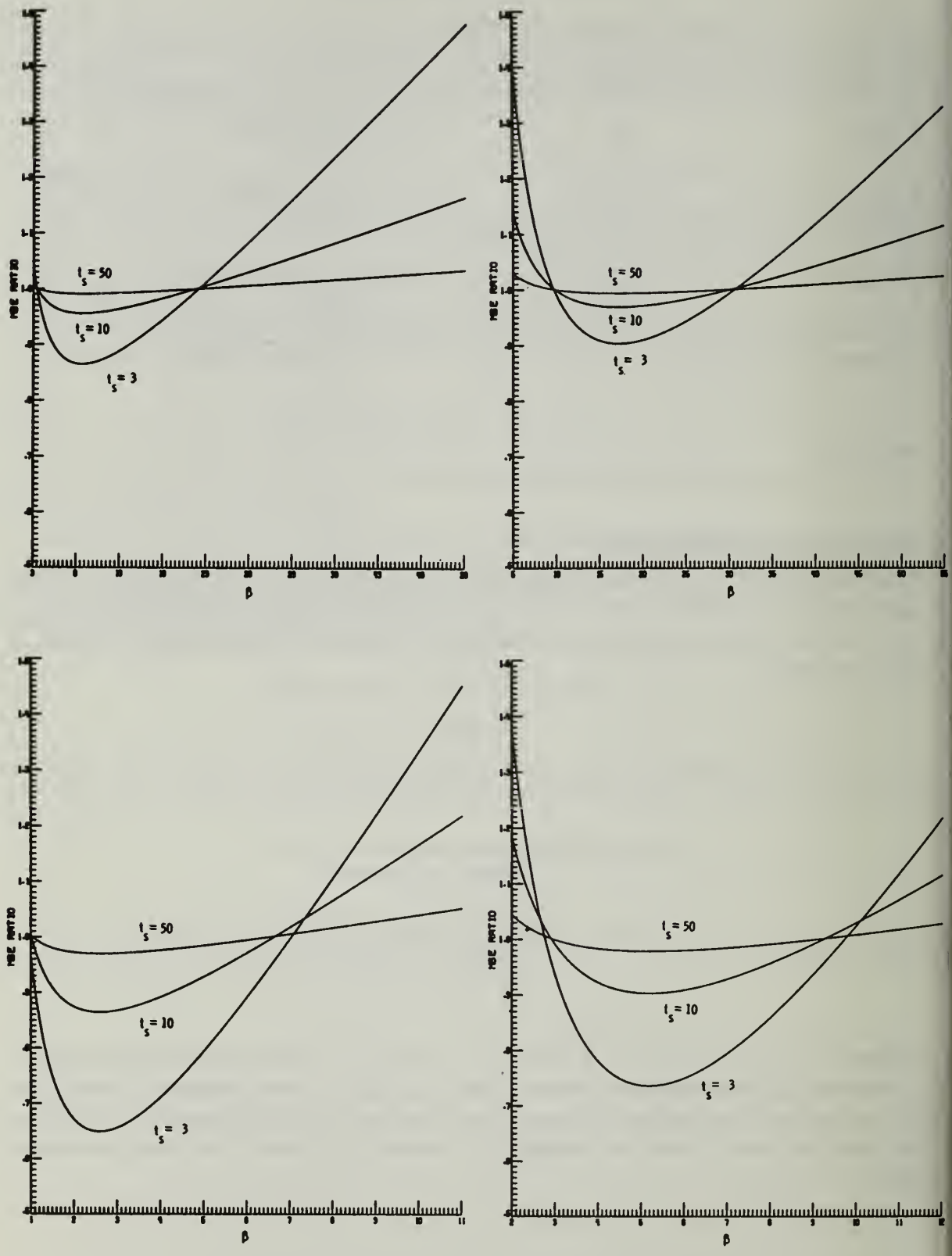


FIGURE 1. Ratio of risk of $\delta(x)$ to MSE of $\hat{\theta}$ when true prior is uniform.

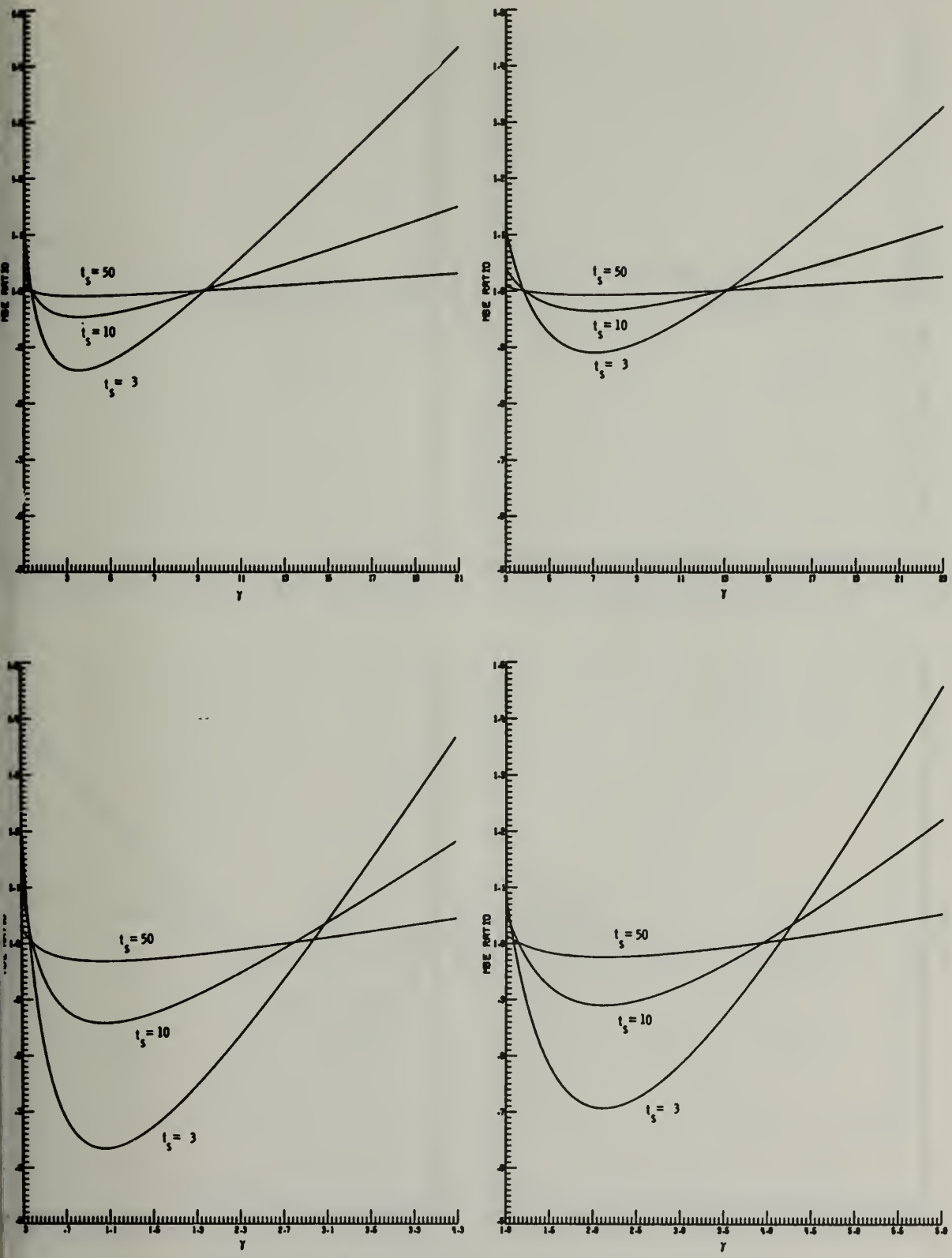


FIGURE 2. Ratio of risk of $\delta(x)$ to MSE of $\hat{\theta}$ when true prior is Rayleigh.

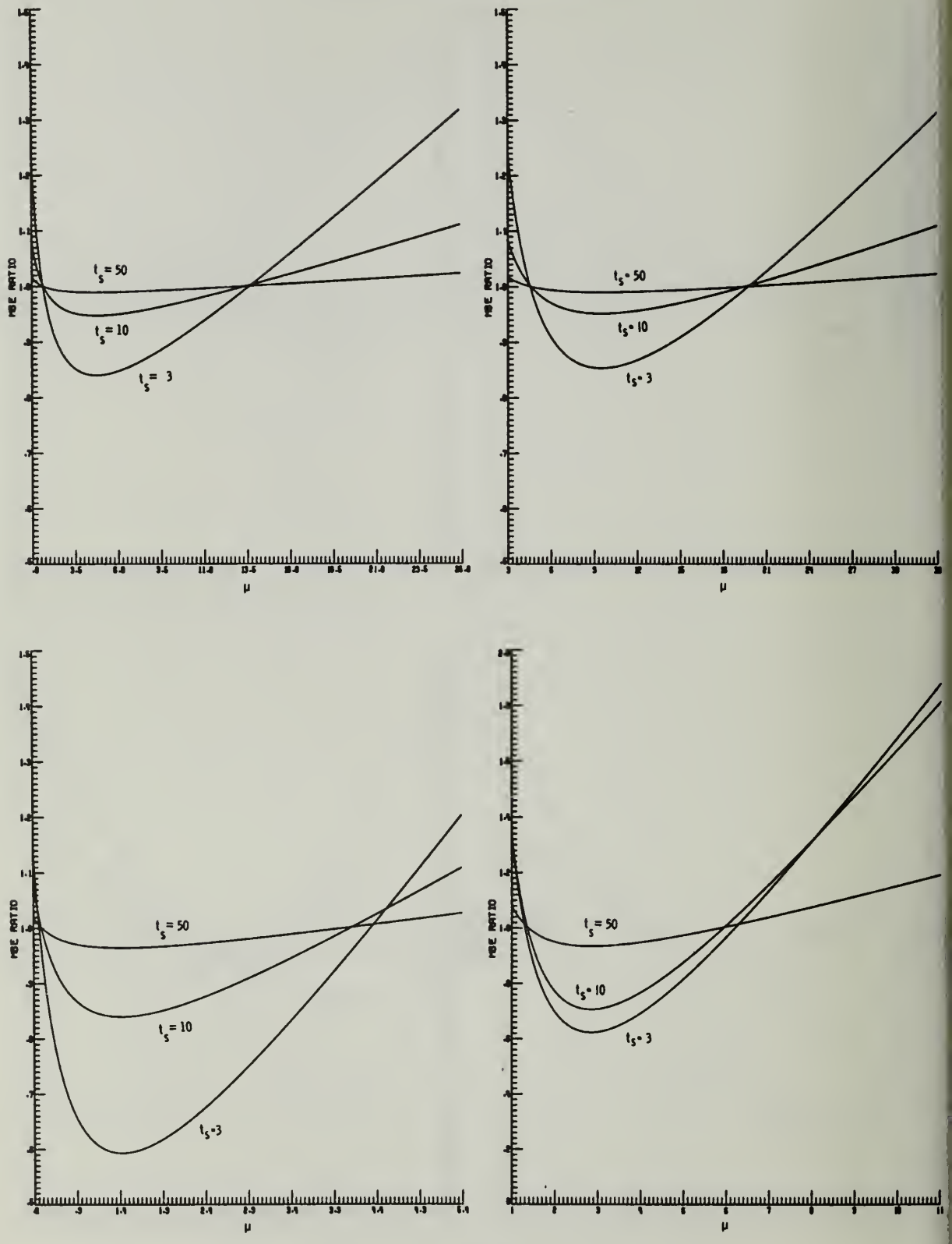


FIGURE 3. Ratio of risk of $\delta(x)$ to MSE of $\hat{\theta}$ when true prior is normal.

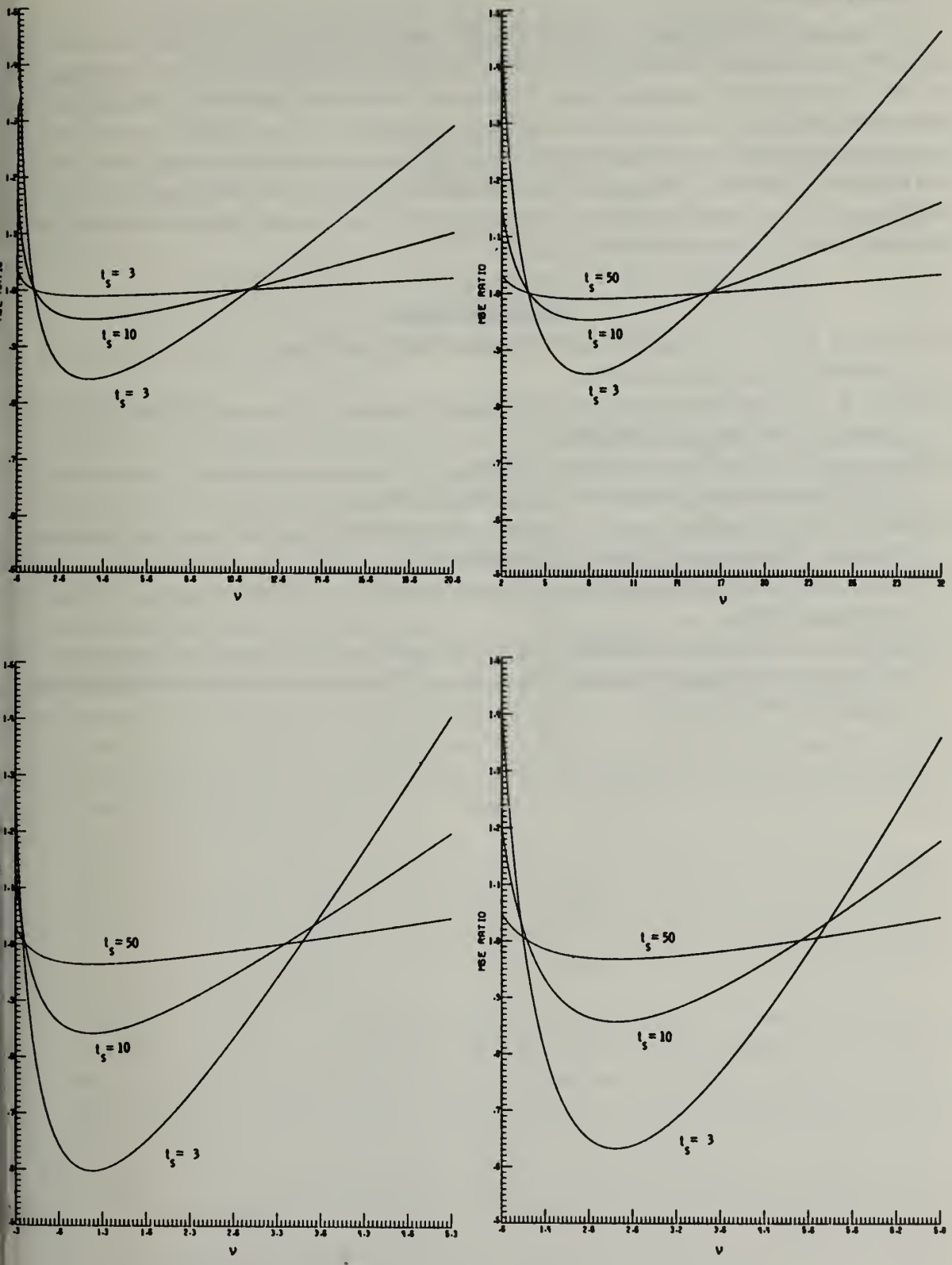


FIGURE 4. Ratio of risk of $\delta(x)$ to MSE of $\hat{\theta}$ when true prior is extreme value.

## 5. CONCLUSIONS

It is apparent from the figures that even though there is a discrepancy between assigned and true priors, the Bayes decision function of the intensity parameter maintains squared error advantage over the MVU estimator for a significant range of values of the true prior parameters. The smallest ratio occurs when the means of the assigned and true prior distributions are essentially the same. Thus, it is reasonable to conclude here that the risk of the Bayes estimator is more sensitive to the central tendency of $\theta$ than it is to the form of the assigned prior distribution. That is to say, knowledge of only a broad neighborhood of the central tendency is sufficient for the Bayes decision function to maintain MSE advantage over the corresponding MVU type.

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# SIMULATING STABLE STOCHASTIC SYSTEMS, V: COMPARISON OF RATIO ESTIMATORS*

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**ABSTRACT**

The regenerative method for estimating parameters in a simulation requires the simulator to estimate the ratio of two means. Five point estimates and four confidence intervals for this ratio have been computed for three stochastic simulations. The jackknife method appears to be the most promising for both point and interval estimation.

## 1. INTRODUCTION

Simulators are frequently faced with the problem of estimating a parameter which is associated with the stationary (steady-state) distribution of a stochastic process being simulated. A methodology based on regenerative processes for obtaining confidence intervals for such parameters is developed in the first three papers of this series. The crucial requirements for applying this regenerative method are that the process being simulated return infinitely often to a fixed state, the expected time between returns be finite, and the process start from scratch each time it enters the fixed state.

In [3] the problem of estimating the steady-state of stable queueing systems is addressed. A typical problem is to estimate the expected value of the stationary waiting time; here the fixed return state for the waiting time process is 0. Markov chains in discrete and continuous time are discussed in [4]. A typical problem in this case is to estimate the expected number of machines down in the steady-state for the classical repairman problem; the fixed return state here can be taken to be the state in which no machines are down. Finally, in [5] the method is extended to regenerative processes, which are shown to include many discrete-event simulations. For a complete description of the regenerative method the reader is referred to [3], [4], and [5].

While the regenerative method mentioned above appears to be attractive when it can be applied, there will be other simulations in which the return state structure is missing: the process being simulated may converge to a steady-state without returning to a fixed state infinitely often, or the returns may be few and far between. In [6] we present a number of approximation techniques which may be useful in circumventing this difficulty.

Our concern in this paper is in evaluating a variety of point and interval estimators which can be used in conjunction with the regenerative method. The basic problem in statistical inference is to obtain point estimates and confidence intervals for the constant $r=E\{f(X)\}$, where $f$ is a general real-valued function and $X$ is the stationary random variable (r.v.) associated with the process being simulated. In the regenerative method, we observe the process being simulated, $\{X(s) : s \geq 0\}$, in cycles of lengths $\alpha_1, \alpha_2, \ldots, \alpha_n$ and record in each cycle the values $Y_1, Y_2, \ldots, Y_n$, where $Y_i$ is the area under the curve $f(X(s))$ in the $i$th cycle. The crucial facts that enable the regenerative method

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to be used are that $r=EY_1/E\alpha_1$ and that the $n$ pairs $(Y_1, \alpha_1), (Y_2, \alpha_2), \ldots, (Y_n, \alpha_n)$ are independent and identically distributed (i.i.d.). Let $\mathbf{U}=(Y_i, \alpha_i)$ be a column vector with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$; denote the sample mean by

$$\bar{\mathbf{U}}=\begin{pmatrix}\bar{Y}\\ \bar{\alpha}\end{pmatrix}=\frac{1}{n}\sum_{i=1}^{n}\mathbf{U}_i$$

and the sample covariance by

$$\mathbf{s}=\begin{pmatrix}s_{11} & s_{12}\\ s_{12} & s_{22}\end{pmatrix}=\frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{U}_i-\bar{U})(\mathbf{U}_i-\bar{U})',$$

where the prime is for transpose. Suppose we are interested in $100(1-\gamma)$ percent confidence intervals for $r$, and let $k=[\Phi^{-1}(1-\gamma/2)]^2/n$, where $\Phi$ is the mean zero, variance one normal distribution function. The five point estimators of $r$ we consider are as follows:

(1) *Fieller estimator* $$\hat{r}_f(n)=\frac{\bar{Y}\bar{\alpha}-ks_{12}}{\bar{\alpha}^2-ks_{22}};$$

(2) *Beale estimator* $$\hat{r}_b(n)=\frac{\bar{Y}}{\bar{\alpha}}\cdot\frac{(1+s_{12}/n\bar{Y}\bar{\alpha})}{(1+s_{22}/n\bar{\alpha}^2)};$$

(3) *Classical estimator* $$\hat{r}_c(n)=\frac{\bar{Y}}{\bar{\alpha}};$$

(4) *Jackknife estimator* $$\hat{r}_j(n)=\frac{1}{n}\sum_{i=1}^{n}\theta_i,$$

where $\theta_i=n(\bar{Y}/\bar{\alpha})-(n-1)\left(\sum_{j\neq i}Y_j\Big/\sum_{j\neq i}\alpha_j\right)$; and the

(5) *Tin estimator* $$\hat{r}_t(n)=\frac{\bar{Y}}{\bar{\alpha}}\left[1+\left(\frac{s_{12}}{\bar{Y}\bar{\alpha}}-\frac{s_{22}}{\bar{\alpha}^2}\right)n^{-1}\right]$$

The Fieller estimator is the midpoint of the confidence interval proposed by Fieller [8] and treated by Roy and Potthoff [13]; it should be mentioned that Fieller did not propose that it be used as a point estimator. The Beale estimator was first suggested in Beale [1], the classical estimator has undoubtedly been around for years, and the Tin estimator first proposed by Tin [14]. The jackknife estimator is an outgrowth of work by Quenouille [11, 12]; for a current review of the jackknife method consult Miller [10], which also contains a list of references dealing with ratio estimators. We note in passing that all four estimators are strongly consistent (converge to $r$ with probability one) and generally are biased. The Beale, jackknife, and Tin estimators were all concocted with the aim of reducing the bias of the classical estimator.

The problem of constructing confidence intervals for $r$ can be approached in a number of ways. As a first approach let $Z_i = Y_i - r\alpha_i$. Note that the $Z_i$'s are i.i.d. since the $\underset{\sim}{U}_i$'s were. Also observe that $EZ_i = 0$ and $\sigma^2(Z_i) = \sigma_{11} - 2r\sigma_{12} + r^2\sigma_{22} \equiv \sigma^2$. Assuming (as we do) that $0 < \sigma^2 < \infty$, the central limit theorem tells us that

$$\sum_{i=1}^{n} Z_i/\sigma n^{1/2} \Rightarrow N(0,1) \qquad \text{as } n \to \infty, \tag{6}$$

where $N(0, 1)$ stands for a mean zero, variance one normal r.v. and $\Rightarrow$ denotes weak convergence. Rewriting (6) we see that

$$\frac{n^{1/2}[\hat{r_c}(n) - r]}{(\sigma/\bar{\alpha})} \Rightarrow N(0,1) \qquad \text{as } n \to \infty. \tag{7}$$

Since $\bar{\alpha} \to E\alpha$ a.s., we could replace $\bar{\alpha}$ by $E\alpha$ in (7) and not change the result. Similarly, since $n^{1/2}(\hat{r_c}(n) - \hat{r_b}(n)) \to 0$ and $n^{1/2}(\hat{r_c}(n) - \hat{r_t}(n)) \to 0$ a.s. we could also replace $\hat{r_c}(n)$ in (7) by either $\hat{r_b}(n)$ or $\hat{r_t}(n)$ and not change the result. In simulations one can very rarely compute $\sigma$. Even in cases where it has been computed it is a difficult task. Thus it is evident that the principle task in deriving confidence intervals for $r$ is to estimate $\sigma$. The Fieller method for obtaining confidence intervals which was used in [4], [5], and [6] results from estimating $\sigma$ in (7) by $(s_{11} - 2rs_{12} + r^2 s_{22})^{1/2}$ and then solving a quadratic inequality in $r$ to obtain the $100(1-\gamma)$ percent confidence interval

$$\hat{I_f}(n) = \left[ \hat{r_f} - \frac{D^{1/2}}{(\bar{\alpha}^2 - ks_{22})}, \hat{r_f} + \frac{D^{1/2}}{(\bar{\alpha}^2 - ks_{22})} \right], \tag{8}$$

where $D = (\bar{Y}\bar{\alpha} - ks_{12})^2 - (\bar{\alpha}^2 - ks_{22})\ (\bar{Y}^2 - ks_{11})$. The classical method estimates $\sigma$ by $(s_{11} - 2\hat{r_c}s_{12} + \hat{r_c}^2 s_{22})^{1/2} \equiv \hat{s}_c(n)$. Combining this estimate for $\sigma$ together with the classical point estimator for $r$ yields a $100\,(1-\gamma)$ percent confidence interval

$$\hat{I_c}(n) = [\, \hat{r_c} - z_{1-\gamma/2}\hat{s_c}/\bar{\alpha}\, n^{1/2},\ \hat{r_c} + z_{1-\gamma/2}\hat{s_c}/\bar{\alpha}\, n^{1/2}\,], \tag{9}$$

where $z_{1-\gamma=2} = \Phi^{-1}(1-\gamma/2)$.

The third type of confidence interval considered comes from the jackknife method. In terms of the notation introduced above let $\hat{s}_j(n) = \left\{ \sum_{i=1}^{n} [\theta_i - \hat{r}_j(n)]^2/(n-1) \right\}^{1/2}$. Then the following limit theorem provides a basis for our confidence interval:

$$n^{1/2}[\hat{r_j}(n) - r]/\hat{s_j} \Rightarrow N(0,1) \qquad \text{as } n \to \infty.$$

Thus the jackknife method yields the $100\ (1-\gamma)$ percent confidence interval

$$\hat{I_j}(n) = [\hat{r_j} - z_{1-\gamma/2}\hat{s_j}/n^{1/2},\ \hat{r_j} + z_{1-\gamma/2}\hat{s_j}/n^{1/2}]. \tag{10}$$

As mentioned above, the four point estimators given in (1)–(4) are asymptotically equivalent as $n \to \infty$

in the sense of strong consistency. The variance estimates that enter in the confidence intervals (8)–(10) also converge a.s. as $n \to \infty$ to $\sigma/E\alpha$. Furthermore, it is known at least in the parametric case where $(Y_i, \alpha_i)$ has a bivariate normal distribution that the point estimators (1)–(5) are asymptotically minimum bound estimators in the sense of the Cramér-Rao inequality; see Tin [14] for this result. Thus one would expect the five point estimators, (1)–(5), and the three confidence intervals (8)–(10) to yield comparable results for large values of $n$.

In an attempt to provide some guidance for simulators we have evaluated the performance of these estimators on three simple stochastic models: the $M/M/1$ queue with $\rho = 0.5$, the classical repairman model, and an $(s, S)$ inventory model. Our objective is to compare both the small and large sample properties of the point and interval estimators described above. While it may be dangerous to extrapolate experimental results from these three models to simulations in general, a number of observations have emerged which we feel will be useful to other simulators.

This paper is organized in the following manner. Section 2 contains a precise description of the three stochastic models simulated and the random number generator used. In Section 3 the experimental data from the simulations is presented. In Section 4 conclusions are drawn from the data. Finally, in Section 5 some recommendations to simulators are given.

## 2. STOCHASTIC MODELS AND RANDOM NUMBER GENERATOR

The first model considered is the $M/M/1$ queue in light traffic; see [3] for further discussion. In this model customer number 0 arrives at time $t_0 = 0$, finds the server free, and experiences an exponentially distributed service time $v_0$ having parameter $\mu > 0$ (the service rate). The $n$th customer arrives at $t_n$ and (eventually) experiences a service time $v_n$ (not including his waiting time). All the $v_n$'s, $n \geq 0$, are independent and have the same distribution as $v_0$. The interarrival times $u_n = t_n - t_{n-1}$, $n \geq 1$, are independent and have an exponential distribution with parameter $\lambda > 0$ (the arrival rate). Furthermore, the two sequences $\{v_n : n \geq 0\}$ and $\{u_n : n \geq 1\}$ are independent. The natural measure of congestion for this system is the traffic intensity $\rho = \lambda/\mu$. We simulate the stochastic process $\{W_n : n \geq 0\}$, where $W_n$ is the waiting time (exclusive of service time) of the $n$th customer. If we define $X_n = v_{n-1} - u_n$, then $W_n$ can be defined recursively by the relations

$$W_0 = 0,$$

$$W_{n+1} = [W_n + X_{n+1}]^+, \qquad n \geq 0, \tag{11}$$

where $[x]^+ = \max(0, x)$.

Observe that $\{W_n : n \geq 0\}$ is a discrete time Markov process with state space $[0, \infty)$. It is well-known that $W_n \Rightarrow W$, a nondegenerate random variable, if and only if $\rho < 1$. The closer $\rho$ is to one the slower the rate of convergence and the longer are the simulation runs required for fixed precision. In all our simulations $\lambda = 5$, $\mu = 10$, and $\rho = 0.5$. We restrict our attention to estimating $EW = \rho/(1-\rho)\mu = 0.1$. In this case $\alpha_n$ is the number of customers served in the $n$th busy period and $Y_n$ is the sum of the waiting times, $W_k$'s, in the $n$th busy period. Furthermore, $Z_n = Y_n - (EW)\alpha_n$. Also we should mention that $E\alpha_1 = 1/(1-\rho) = 2$ for $\rho = 0.5$ and that the value of $\sigma^2$ has been calculated by Blomqvist [2] and Dailey [7]:

$$\sigma^2 = EZ_1^2 = E\alpha_1 \left[ \frac{4\rho}{(1-\rho)^4 \mu^2} - \frac{\rho(2-\rho)}{(1-\rho)^2 \mu^2} \right],$$

which yields $\sigma^2 = 0.58$ for $\rho = 0.5$. Thus $\sigma/E\alpha = 0.3808$.

The second stochastic model considered is a periodic review inventory model. The inventory system is operated under a stationary $(s, S)$ policy. Let $X_n$ denote the level of inventory on hand plus on order in period $n$ immediately after the ordering decision and let $d_n$ denote the demand in period $n$. Then we have the recursion

$$X_{n+1} = \begin{cases} X_n - d_n, & d_n \leq X_n - s \\ S, & \text{otherwise.} \end{cases}$$

If we set $X_0 = S$, then $s \leq X_n \leq S$ for all $n \geq 0$. For our simulation we have taken the demand distribution to be described by $P\{d_n = k\} = 1/3$, $(k = 0, 1, 2)$, set $s = 3$ and $S = 6$, and used $S$ as the return state for the regenerative method. Hence we have a simple four state discrete time Markov chain. Again $X_n \Rightarrow X$ and we estimate $EX = 4.653$. In this case $\sigma/E\alpha = 0.577$. For further discussion of this model see Wagner [15] (p. A44).

Finally, our third model is the classical repairman problem which is a continuous time Markov chain. We have $m+n$ identical pieces of equipment, each with an exponential failure time having parameter $\lambda$. At most $n$ of the units operate at one time, the other $m$ units being thought of as spares. When a unit fails, it is sent to a repair facility consisting of $s$ repairmen (servers) having exponential repair (service) times with parameter $\mu$. Let $X(t)$ denote the number of failed units undergoing or waiting for service at the repair facility at time $t$. With the above assumptions $\{X(t) : t \geq 0\}$ is a birth-death process with state space $\{0, 1, \ldots, m+n\}$, birth parameters $\lambda_i = \lambda(n - [i-m]^+)$, and death parameters $\mu_i = \mu(i \wedge s)$ where $a \wedge b = \min(a, b)$. We have set $X(0) = 0$ and used 0 as the return state for the regenerative method. For parameter values we have taken $n = 10$, $m = 4$, $s = 3$, $\lambda = 1$, and $\mu = 4$. Again $X(t) \Rightarrow X$ and we estimate $EX = 3.471$. Here $\sigma/E\alpha = 1.9553$.

The random number generator (r.n.g.) used in all simulations was one written by G. E. Forsythe following the discussion in Knuth [9] (p. 155–156) for use on the IBM 360/67. First a sequence of random integers $\{X_n : n \geq 0\}$ uniformly distributed on $[0, 2^{31})$ is generated as follows: $X_{n+1} = [(314159269)X_n + 453806245] \bmod (2^{31})$, $n \geq 0$, where $X_0$ is a given seed. Then a sequence of uniformly distributed numbers, $\{Y_n : n \geq 1\}$, is obtained by setting $Y_n = X_n/2^{31}$.

A word of caution is in order with respect to the use of the r.n.g.'s in the models described above. In the $M/M/1$ queue the simulator needs to generate two i.i.d. sequences of exponential r.v.'s, $\{v_n : n \geq 0\}$ and $\{u_n : n \geq 1\}$, which are themselves independent. This can be done using either one or two r.n.g.'s. The results obtained are slightly different, presumably because the use of one r.n.g. might introduce some serial correlation into the sequence $v_0, u_1, v_1, u_2, \ldots$. While this aspect of simulation is still somewhat of a mystery, we recommend the use of two independent r.n.g.'s. The $M/M/1$ simulations to be reported in the next section used two r.n.g.'s.

## 3. EXPERIMENTAL RESULTS

In this section we shall summarize the simulation results for the three stochastic models described in Section 2. Each model was run for a certain number of cycles and observations taken at periodic

intervals. Then the runs were replicated. The first table for each model gives the mean (over the various replications) of the five point estimates (1)–(5), as a function of the number of cycles. Confidence intervals for these means are also indicated. All confidence intervals are taken to have 90-percent degree of confidence. The second table will give the mean (over the replications) and confidence intervals of estimates of $\sigma/E\alpha$ generated by the Fieller, jackknife, and classical methods; namely, $(nD)^{1/2}/(\bar{\alpha}^2 - ks_{22})z_{0.95}$, $\hat{s}_j$, and $\hat{s}_c/\bar{\alpha}$. The final table will give the fraction of replications for which the confidence intervals given by (8)–(10) and a combination of the Tin point estimator with the classical variance estimator cover the true parameter being estimated. Again confidence intervals for these coverage variables are indicated. In addition, for the $M/M/1$ model the coverage of confidence intervals based on the Beale point estimator and the known value of $\sigma/E\alpha$ are given.

For the $M/M/1$ model we have taken $\lambda = 5$ and $\mu = 10$ giving rise to $\rho = 0.5$. The small sample runs ($\leq$ 400 cycles) were replicated 200 times while the large sample runs ($>$ 400 cycles) were replicated 100 times. It may seem like 400 cycles is a large sample, however, one-half the length of confidence intervals produced from this number of cycles is only 30-percent of the true value of $EW = 0.1$. To obtain a confidence interval with half length 10-percent of the true value of $EW$ requires roughly 3,900 cycles based on the known value of $\sigma/E\alpha$. The results for our longest runs (2,000 cycles) produce confidence intervals whose half length is 14-percent of the true $EW$. Tables 1–3 summarize the experimental results for the $M/M/1$ queue. For all tables the upper number in a given box is the average over the 100 or 200 replications of the quantity in question, while the lower figure is one-half the length of a 90-percent confidence interval about the mean. By way of example, for 80 cycles a 90-percent confidence interval for the mean of the jackknife point estimator is $[0.0990-0.0050, 0.0990+0.0050]$.

TABLE 1. *Mean and Confidence Intervals of Point Estimators for* EW = 0.1 *in an* M/M/1 *Queue with* $\rho = 0.5$

| Number of cycles | Point estimator of $EW$ | | | | |
|---|---|---|---|---|---|
| | Fieller | Jackknife | Beale | Tin | Classical |
| 40 | 0.0749<br>±0.0040 | 0.0989<br>±0.0067 | 0.0975<br>±0.0064 | 0.0978<br>±0.0065 | 0.0923<br>±0.0059 |
| 80 | 0.0849<br>±0.0035 | 0.0990<br>±0.0050 | 0.0985<br>±0.0049 | 0.0986<br>±0,0049 | 0.0951<br>±0.0045 |
| 120 | 0.0894<br>±0.0031 | 0.0992<br>±0.0040 | 0.0990<br>±0.0040 | 0.0990<br>±0.0040 | 0.0965<br>±0.0038 |
| 160 | 0.0909<br>±0.0028 | 0.0982<br>±0.0033 | 0.0981<br>±0.0033 | 0.0981<br>±0.0033 | 0.0962<br>±0.0032 |
| 200 | 0.0922<br>±0.0026 | 0.0981<br>±0.0033 | 0.0980<br>±0.0029 | 0.0980<br>±0.0029 | 0.0965<br>±0.0028 |
| 400 | 0.0967<br>±0.0021 | 0.0999<br>±0.0022 | 0.0998<br>±0.0022 | 0.0998<br>±0.0022 | 0.0990<br>±0.0022 |
| 800 | 0.0978<br>±0.0021 | 0.0993<br>±0.0022 | 0.0993<br>±0.0022 | 0.0993<br>±0.0022 | 0.0989<br>±0.0022 |
| 1200 | 0.0992<br>±0.0017 | 0.1003<br>±0.0017 | 0.1003<br>±0.0017 | 0.1003<br>±0.0017 | 0.1000<br>±0.0017 |
| 1600 | 0.0995<br>±0.0015 | 0.1003<br>±0.0015 | 0.1003<br>±0.0015 | 0.1003<br>±0.0015 | 0.1001<br>±0.0015 |
| 2000 | 0.0994<br>±0.0015 | 0.1000<br>±0.0015 | 0.1000<br>±0.0015 | 0.1000<br>±0.0015 | 0.0999<br>±0.0015 |

TABLE 2. *Mean and Confidence Intervals of Point Estimators for* $\sigma/E\alpha = 0.3808$ *in an* M/M/1 *Queue with* $\rho = 0.5$

| Number of cycles | Point estimator of $\sigma/E\alpha$ | | |
|---|---|---|---|
| | Fieller | Jackknife | Classical |
| 40 | 0.2358 ±0.0210 | 0.2456 ±0.0223 | 0.2112 ±0.0167 |
| 80 | 0.2770 ±0.0234 | 0.2889 ±0.0256 | 0.2624 ±0.0206 |
| 120 | 0.2946 ±0.0216 | 0.3055 ±0.0235 | 0.2847 ±0.0202 |
| 160 | 0.3028 ±0.0202 | 0.3122 ±0.0217 | 0.2955 ±0.0193 |
| 200 | 0.3065 ±0.0185 | 0.3145 ±0.0197 | 0.3009 ±0.0179 |
| 400 | 0.3414 ±0.0175 | 0.3469 ±0.0183 | 0.3384 ±0.0173 |
| 800 | 0.3441 ±0.0189 | 0.3469 ±0.0193 | 0.3427 ±0.0188 |
| 1200 | 0.3591 ±0.0157 | 0.3612 ±0.0159 | 0.3581 ±0.0156 |
| 1600 | 0.3675 ±0.0132 | 0.3692 ±0.0133 | 0.3667 ±0.0131 |
| 2000 | 0.3685 ±0.0147 | 0.3699 ±0.0149 | 0.3679 ±0.0147 |

TABLE 3. *Mean and Confidence Intervals of Coverage of Various 90 Percent Confidence Intervals for* EW = 0.1 *in an* M/M/1 *Queue with* $\rho = 0.5$

| Number of Cycles | Confidence interval* for EW | | | | |
|---|---|---|---|---|---|
| | Fieller / Fieller | Jackknife / Jackknife | Tin / Classical | Classical / Classical | Beale / Known |
| 40 | 0.6150 ±0.0567 | 0.6950 ±0.0537 | 0.6500 ±0.0556 | 0.6550 ±0.0554 | 0.9500 ±0.0254 |
| 80 | 0.6650 ±0.0550 | 0.7350 ±0.0515 | 0.7100 ±0.0529 | 0.6950 ±0.0537 | 0.9350 ±0.0287 |
| 120 | 0.7500 ±0.0505 | 0.7850 ±0.0479 | 0.7750 ±0.0487 | 0.7650 ±0.0494 | 0.9250 ±0.0307 |
| 160 | 0.7650 ±0.0494 | 0.7900 ±0.0475 | 0.7800 ±0.0483 | 0.7700 ±0.0491 | 0.9300 ±0.0298 |
| 200 | 0.7700 ±0.0491 | 0.7950 ±0.0471 | 0.7950 ±0.0471 | 0.7950 ±0.0471 | 0.9300 ±0.0298 |
| 400 | 0.8100 ±0.0475 | 0.8150 ±0.0453 | 0.8100 ±0.0457 | 0.8100 ±0.0457 | 0.9050 ±0.0342 |
| 800 | 0.8300 ±0.0621 | 0.8300 ±0.0621 | 0.8300 ±0.0621 | 0.8300 ±0.0621 | 0.9000 ±0.0496 |
| 1200 | 0.9000 ±0.0496 | 0.9100 ±0.0473 | 0.9100 ±0.0473 | 0.9100 ±0.0473 | 0.9100 ±0.0473 |
| 1600 | 0.8500 ±0.0590 | 0.8500 ±0.0590 | 0.8500 ±0.0590 | 0.8500 ±0.0590 | 0.9100 ±0.0473 |
| 2000 | 0.8200 ±0.0635 | 0.8300 ±0.0621 | 0.8300 ±0.0621 | 0.8200 ±0.0635 | 0.8500 ±0.0590 |

*Confidence interval $A/B$ is centered about the point estimator $A$ and uses method $B$ to estimate $\sigma/E\alpha$.

Additional data collected were estimates for $E\alpha$, $EY$, $\sigma_{11}$, $\sigma_{12}$, $\sigma_{22}$, and $\sigma_{12}/(\sigma_1, \sigma_{12})^{1/2}$. For 2,000 cycles the means over 100 replications of $\hat{\alpha}$, $\hat{Y}$, $\hat{s}_{11}$, $\hat{s}_{12}$, $\hat{s}_{22}$, and $\hat{s}_{12}/(\hat{s}_{11}\hat{s}_{22})^{1/2}$ were

$$\bar{\hat{\alpha}} = 1.9934\ (0.0569), \quad \bar{\hat{Y}} = 0.1995\ (0.8643), \quad \bar{\hat{s}}_{11} = 0.9179\ (0.4636), \quad \bar{\hat{s}}_{12} = 1.9781\ (0.5619),$$

$$\bar{\hat{s}}_{22} = 5.8927\ (0.9482), \quad \text{and } \overline{\hat{s}_{12}/(\hat{s}_{11}\hat{s}_{12})^{1/2}} = 0.8643\ (0.0265),$$

where the figures in parentheses are the sample standard deviations of the 100 numbers averaged.

Tables 4–6 contain the experimental results for the $(s, S)$ inventory model with $s=3$ and $S=6$. The small runs ($\leq 40$ cycles) were replicated 200 times and the large sample ($>40$ cycles) 100 times. Four cycles produces confidence intervals whose half length is 11 percent of the true value of $EX=4.653$. While this seems quite good, it is somewhat illusory since the central limit theorem is not a good approximation for $n=4$. Our longest runs of 100 cycles yield confidence intervals whose half length is 2 percent of $EX$. For 100 cycles the means over 100 replications of $\hat{\alpha}$, $\hat{Y}$, $\hat{s}_{11}$, $\hat{s}_{12}$, $\hat{s}_{22}$, and $\hat{s}_{12}/(\hat{s}_{11}\hat{s}_{12})^{1/2}$ were

$$\bar{\hat{\alpha}} = 2.8632\ (0.1747), \quad \bar{\hat{Y}} = 13.3373\ (0.6924), \quad \bar{\hat{s}}_{11} = 50.6844\ (9.5535), \quad \bar{\hat{s}}_{12} = 12.5759\ (2.2691),$$

$$\bar{\hat{s}}_{22} = 3.1892\ (0.5627), \quad \text{and } \overline{\hat{s}_{12}/(\hat{s}_{11}\hat{s}_{12})^{1/2}} = 0.9895\ (0.0036).$$

TABLE 4. *Mean and Confidence Interval of Point Estimators for* EX = 4.653 *in an* (s, S) *Inventory Model with* s = 3, S = 6

| Number of cycles | Point estimator of $EX$ | | | | |
|---|---|---|---|---|---|
| | Fieller | Jackknife | Beale | Tin | Classical |
| 4 | 5.0866 | 4.6630 | 4.6878 | 4.6798 | 4.7536 |
| | ±0.0683 | ±0.0375 | ±0.0374 | ±0.0374 | ±0.0383 |
| 8 | 4.8226 | 4.6597 | 4.6654 | 4.6634 | 4.7010 |
| | ±0.0310 | ±0.0258 | ±0.0259 | ±0.0259 | ±0.0265 |
| 12 | 4.7567 | 4.6569 | 4.6590 | 4.6581 | 4.6828 |
| | ±0.0220 | ±0.0206 | ±0.0206 | ±0.0206 | ±0.0207 |
| 20 | 4.7153 | 4.6588 | 4.6595 | 4.6593 | 4.6737 |
| | ±0.0161 | ±0.0153 | ±0.0153 | ±0.0153 | ±0.0154 |
| 40 | 4.6867 | 4.6598 | 4.6600 | 4.6599 | 4.6670 |
| | ±0.0112 | ±0.0110 | ±0.0109 | ±0.0109 | ±0.0110 |
| 60 | 4.6857 | 4.6681 | 4.6682 | 4.6682 | 4.6728 |
| | ±0.0140 | ±0.0138 | ±0.0138 | ±0.0138 | ±0.0139 |
| 70 | 4.6807 | 4.6655 | 4.6655 | 4.6655 | 4.6695 |
| | ±0.0130 | ±0.0128 | ±0.0128 | ±0.0128 | ±0.0129 |
| 80 | 4.6732 | 4.6599 | 4.6599 | 4.6599 | 4.6635 |
| | ±0.0117 | ±0.0115 | ±0.0116 | ±0.0117 | ±0.0116 |
| 90 | 4.6700 | 4.6583 | 4.6583 | 4.6583 | 4.6614 |
| | ±0.0109 | ±0.0109 | ±0.0109 | ±0.0109 | ±0.0108 |
| 100 | 4.6686 | 4.6581 | 4.6582 | 4.6581 | 4.6609 |
| | ±0.0104 | ±0.0102 | ±0.0102 | ±0.0102 | ±0.0104 |

TABLE 5. *Mean and Confidence Interval of Point Estimators for* $\sigma/E\alpha = 0.5771$ *in an* (s, S) *Inventory Model with* $s = 3$, $S = 6$

| Number of cycles | Point estimator of $\sigma/E\alpha$ | | |
|---|---|---|---|
| | Fieller | Jackknife | Classical |
| 4 | 0.7804 | 0.6287 | 0.5296 |
| | ±0.0718 | ±0.0416 | ±0.0263 |
| 8 | 0.6489 | 0.6157 | 0.5648 |
| | ±0.0300 | ±0.0280 | ±0.0205 |
| 12 | 0.6149 | 0.5971 | 0.5656 |
| | ±0.0189 | ±0.0190 | ±0.0158 |
| 20 | 0.5958 | 0.5876 | 0.5681 |
| | ±0.0139 | ±0.0142 | ±0.0124 |
| 40 | 0.5883 | 0.5798 | 0.5704 |
| | ±0.0106 | ±0.0108 | ±0.0101 |
| 60 | 0.5809 | 0.5787 | 0.5725 |
| | ±0.0118 | ±0.0119 | ±0.0114 |
| 70 | 0.5835 | 0.5817 | 0.5762 |
| | ±0.0113 | ±0.0114 | ±0.0110 |
| 80 | 0.5842 | 0.5827 | 0.5778 |
| | ±0.0105 | ±0.0106 | ±0.0102 |
| 90 | 0.5801 | 0.5787 | 0.5745 |
| | ±0.0103 | ±0.0104 | ±0.0101 |
| 100 | 0.5779 | 0.5767 | 0.5729 |
| | ±0.0099 | ±0.0100 | ±0.0097 |

TABLE 6. *Mean and Confidence Intervals of Coverage of Various 90 Percent Confidence Intervals for* EX *in an* (s, S) *Inventory Model with* $s = 3$, $S = 6$

| Number of cycles | Confidence interval* for *EX* | | | |
|---|---|---|---|---|
| | Fieller / Fieller | Jackknife / Jackknife | Tin / Classical | Classical / Classical |
| 4 | 0.7100 | 0.8350 | 0.8050 | 0.7850 |
| | ±0.0529 | ±0.0433 | ±0.0462 | ±0.0479 |
| 8 | 0.8250 | 0.8350 | 0.8250 | 0.8300 |
| | ±0.0443 | ±0.0433 | ±0.0443 | ±0.0438 |
| 12 | 0.8300 | 0.8800 | 0.8750 | 0.8500 |
| | ±0.0438 | ±0.0379 | ±0.0386 | ±0.0416 |
| 20 | 0.8800 | 0.9150 | 0.8950 | 0.8950 |
| | ±0.0379 | ±0.0325 | ±0.0357 | ±0.0357 |
| 40 | 0.8650 | 0.8950 | 0.8950 | 0.8800 |
| | ±0.0398 | ±0.0357 | ±0.0357 | ±0.0379 |
| 60 | 0.8700 | 0.8600 | 0.8600 | 0.8600 |
| | ±0.0556 | ±0.0574 | ±0.0574 | ±0.0574 |
| 70 | 0.8500 | 0.8900 | 0.8700 | 0.8700 |
| | ±0.0590 | ±0.0517 | ±0.0556 | ±0.0556 |
| 80 | 0.8700 | 0.8900 | 0.8900 | 0.8900 |
| | ±0.0556 | ±0.0517 | ±0.0517 | ±0.0517 |
| 90 | 0.8700 | 0.9100 | 0.9100 | 0.9100 |
| | ±0.0556 | ±0.0473 | ±0.0473 | ±0.0473 |
| 100 | 0.8900 | 0.8800 | 0.8700 | 0.8900 |
| | ±0.0517 | ±0.0537 | ±0.0556 | ±0.0517 |

*Confidence interval $A/B$ is centered about the point estimator $A$ and uses method $B$ to estimate $\sigma/E\alpha$.

Tables 7–9 contain the experimental results for the repairman model with $n=10$, $m=4$, $s=3$, $\lambda=1$, and $\mu=4$. Again the small sample runs ($\leq 40$ cycles) were replicated 200 times and the large sample ($>40$ cycles) 100 times. For this simulation 4(100) cycles produces confidence intervals whose half length is 46 percent (9 percent) of the true value of $EX=3.471$. For 100 cycles the means over 100 replications of $\hat{\alpha}$, $\hat{Y}$, $\hat{s}_{11}$, $\hat{s}_{12}$, $\hat{s}_{22}$, and $\hat{s}_{12}/(\hat{s}_{11}\hat{s}_{22})^{1/2}$ were

$$\bar{\hat{\alpha}}=1.6618\ (0.2106),\qquad \bar{\hat{Y}}=5.7437\ (0.9471),\qquad \bar{\hat{s}}_{11}=103.4559\ (35.6320),$$

$$\bar{\hat{s}}_{12}=22.2238\ (7.1927),\qquad \bar{\hat{s}}_{22}=5.0540\ (1.5350),\qquad \text{and } \overline{\hat{s}_{12}/(\hat{s}_{11}\hat{s}_{22})^{1/2}}=0.9720\ (0.0103).$$

## 4. CONCLUSIONS FROM EXPERIMENTAL DATA

Suppose we are interested in estimating a parameter $r$ with a $100(1-\gamma)$ percent confidence interval whose half length is $100\delta$ percent of $r$. Then the number of cycles, $n$, required is given roughly by

$$n=\left(\frac{z_{1-\gamma/2}}{\delta}\right)^2\left(\frac{\sigma}{rE\alpha}\right)^2.$$

Thus the value of $(\sigma/rE\alpha)^2$ gives a good measure of the length of simulation run required for a fixed level of accuracy. For the $M/M/1$ queue, $(s, S)$ inventory model, and repairman model the true values of $(\sigma/rE\alpha)^2$ are 14.5, 0.0154, and 0.317, respectively. Thus the $M/M/1$ queue requires runs about 46

TABLE 7. *Mean and Confidence Intervals of Point Estimators for* EX = 3.471 *in a Repairman Model with* n = 10, m = 4. s = 3, $\lambda=1$, $\mu=4$

| Number of cycles | Point estimator of $EX$ | | | | |
|---|---|---|---|---|---|
| | Fieller | Jackknife | Beale | Tin | Classical |
| 4 | 6.8681 | 3.4589 | 3.0391 | 3.0489 | 2.9747 |
| | ±3.7471 | ±0.1673 | ±0.1229 | ±0.1235 | ±0.1206 |
| 8 | −3.9251 | 3.5496 | 3.3737 | 3.3939 | 3.2692 |
| | (*) | ±0.0979 | ±0.0824 | ±0.0836 | ±0.0794 |
| 12 | 3.0112 | 3.5870 | 3.5226 | 3.5481 | 3.4006 |
| | ±0.5291 | ±0.0724 | ±0.0665 | ±0.0679 | ±0.0632 |
| 20 | 3.1368 | 3.5175 | 3.5501 | 3.5748 | 3.4239 |
| | ±0.0524 | ±0.0525 | ±0.0526 | ±0.0536 | ±0.0502 |
| 40 | 3.2977 | 3.4687 | 3.5603 | 3.5860 | 3.4238 |
| | ±0.0340 | ±0.0357 | ±0.0371 | ±0.0379 | ±0.0350 |
| 60 | 3.3475 | 3.4570 | 3.5648 | 3.5905 | 3.4278 |
| | ±0.0376 | ±0.0390 | ±0.0411 | ±0.0424 | ±0.0385 |
| 70 | 3.3576 | 3.4508 | 3.5636 | 3.5894 | 3.4258 |
| | ±0.0348 | ±0.0365 | ±0.0391 | ±0.0404 | ±0.0360 |
| 80 | 3.3747 | 3.4570 | 3.5747 | 3.6013 | 3.4349 |
| | ±0.0315 | ±0.0323 | ±0.0344 | ±0.0355 | ±0.0320 |
| 90 | 3.3782 | 3.4516 | 3.5729 | 3.5999 | 3.4318 |
| | ±0.0311 | ±0.0318 | ±0.0339 | ±0.0349 | ±0.0315 |
| 100 | 3.3933 | 3.4598 | 3.5851 | 3.6122 | 3.4419 |
| | ±0.0309 | ±0.0312 | ±0.0326 | ±0.0334 | ±0.0311 |

*Overflowed the machine.

TABLE 8. *Mean and Confidence Interval of Point Estimators for* $\sigma/E\alpha = 1.9553$ *in a Repairman Model with* $n = 10$, $m = 4$, $s = 3$, $\lambda = 1$, $\mu = 4$

| Number of cycles | Point Estimator of $\sigma/E\alpha$ | | |
|---|---|---|---|
| | Fieller | Jackknife | Classical |
| 4 | −4.4548 | 1.8404 | 0.9368 |
| | ±4.5332 | ±0.1607 | ±0.0544 |
| 8 | 13.4331 | 2.1105 | 1.3815 |
| | ±18.3457 | ±0.1533 | ±0.0674 |
| 12 | 2.0163 | 2.1323 | 1.5795 |
| | ±1.157 | ±0.1361 | ±0.0691 |
| 20 | 2.1151 | 1.9769 | 1.6771 |
| | ±0.1036 | ±0.0917 | ±0.0611 |
| 40 | 1.9921 | 1.9652 | 1.8056 |
| | ±0.0635 | ±0.0655 | ±0.0542 |
| 60 | 1.9414 | 1.9336 | 1.8224 |
| | ±0.0818 | ±0.0867 | ±0.0729 |
| 70 | 1.9229 | 1.9165 | 1.8218 |
| | ±0.0720 | ±0.0759 | ±0.0656 |
| 80 | 1.9212 | 1.9174 | 1.8317 |
| | ±0.0667 | ±0.0704 | ±0.0613 |
| 90 | 1.9238 | 1.9213 | 1.8438 |
| | ±0.0654 | ±0.0687 | ±0.0605 |
| 100 | 1.9513 | 1.9508 | 1.8791 |
| | ±0.0615 | ±0.0645 | ±0.0576 |

TABLE 9. *Mean and Confidence Intervals of Coverage of Various 90 percent Confidence Intervals for* EX *in a Repairman Model with* $n = 10$, $m = 4$, $s = 3$, $\lambda = 1$, $\mu = 4$

| Number of cycles | Confidence interval* for *EX* | | | |
|---|---|---|---|---|
| | Fieller / Fieller | Jackknife / Jackknife | Tin / Classical | Classical / Classical |
| 4 | 0.3300 | 0.7050 | 0.5250 | 0.5100 |
| | ±0.0548 | ±0.0532 | ±0.0582 | ±0.0583 |
| 8 | 0.6350 | 0.8050 | 0.7050 | 0.6900 |
| | ±0.0561 | ±0.0462 | ±0.0532 | ±0.0539 |
| 12 | 0.7600 | 0.8500 | 0.7600 | 0.7750 |
| | ±0.0498 | ±0.0416 | ±0.0498 | ±0.0487 |
| 20 | 0.8350 | 0.8550 | 0.7900 | 0.8550 |
| | ±0.0433 | ±0.0411 | ±0.0475 | ±0.0411 |
| 40 | 0.8450 | 0.8600 | 0.8250 | 0.8400 |
| | ±0.0422 | ±0.0405 | ±0.0443 | ±0.0428 |
| 60 | 0.8900 | 0.9000 | 0.8800 | 0.8900 |
| | ±0.0517 | ±0.0496 | ±0.0537 | ±0.0517 |
| 70 | 0.8300 | 0.8700 | 0.8300 | 0.8800 |
| | ±0.0621 | ±0.0556 | ±0.0621 | ±0.0537 |
| 80 | 0.8600 | 0.9100 | 0.8100 | 0.8900 |
| | ±0.0574 | ±0.0473 | ±0.0649 | ±0.0517 |
| 90 | 0.9000 | 0.9000 | 0.7700 | 0.9000 |
| | ±0.0496 | ±0.0496 | ±0.0696 | ±0.0496 |
| 100 | 0.8800 | 0.8900 | 0.7900 | 0.9000 |
| | ±0.0537 | ±0.0517 | ±0.0673 | ±0.0496 |

*Confidence interval $A/B$ is centered about the point estimator $A$ and uses method $B$ to estimate $\sigma/E\alpha$.

times as long as those for the repairman model, which in turn requires runs about 21 times as long as those for the $(s, S)$ inventory model. Note also that $\sigma^2$ contains a term $-2r\sigma_{12}$, so that if $\sigma_{12}$ increases $\sigma^2$ decreases, assuming all other parameters remain fixed. Thus a high value of the correlation coefficient $\rho_0 = \sigma_{12}/(\sigma_{11}\sigma_{22})^{1/2}$ yields a smaller value of $\sigma$, which in turn means the simulation will not require as long runs. Note that our estimate of $\rho_0$ in the $M/M/1$ inventory, and repairman models are 0.8643, 0.9895, and 0.9720, respectively, which is consistent with the above remark.

### Point Estimators of $r$

Of the five point estimators studied the Fieller estimate can be immediately discarded. In all three models it showed considerably more bias than the other estimators for short runs. The jackknife estimator appears to be the best in terms of small bias for short runs. The Beale and Tin estimators were about the same and only slightly more biased than the jackknife estimator. In fourth place comes the classical estimator. For the $M/M/1$ queue and inventory models all five point estimators gave comparable results for long runs. For the repairman model the jackknife and classical estimators gave good results for long runs while the Beale and Tin estimators over estimated the true value of $r$ and the Fieller estimator under estimated $r$.

### Point Estimators of $\sigma/E\alpha$

The estimators of $\sigma/E\alpha$ determine the lengths of the confidence intervals for $r$. Asymptotically for long runs, the Fieller, jackknife, and classical estimators of $\sigma/E\alpha$ performed comparably in the three models. For short runs the jackknife method appears to be best in terms of small bias and most rapid convergence to the true value of $\sigma/E\alpha$. It is about a toss-up between the Fieller and classical estimators for second place. Note that in all three models the estimate for $\sigma/E\alpha$ stabilizes after a moderate number of cycles.

### Coverage of Various Confidence Intervals

Asymptotically for long runs, the Fieller/Fieller, jackknife/jackknife, Tin/classical, and classical/classical confidence intervals give accurate coverage of the parameter $r$ for all models with one exception. The Tin/classical confidence interval for the repairman model does not give adequate coverage, presumably because the Tin point estimator has a significant upward bias. For short runs all four methods produce short confidence intervals which do not adequately cover $r$. This is caused primarily by low estimates of $\sigma/E\alpha$. However, for short runs the jackknife/jackknife method does best followed by the Tin/classical and classical/classical which are about the same. The Fieller/Fieller method comes in last. Note that the Beale/known method produces good results in the $M/M/1$ for short runs. This suggests that the convergence in the central limit theorem (6) is reasonably fast.

## 5. RECOMMENDATIONS

The jackknife method for point and interval estimation of $r$ produced slightly better statistical results than the other methods and is recommended. We point out, however, that the length of confidence intervals produced (relative to $r$) were quite large for those run lengths where the jackknife's superiority was visible. This suggests that the jackknife method is particularly appropriate for any short exploratory runs the simulator might make. Two minor drawbacks of the jackknife method are

a larger memory requirement and slightly more complex programming. Additional storage addresses of the order of $2n$ are required, where $n$ is the number of cycles observed.

If the storage requirements for the jackknife method are excessive, we would recommend using the Beale or Tin methods for point estimates and the classical method for interval estimates. The classical/classical method is the easiest to program and pedagogically the simplest to explain. Thus for the long runs it is an attractive candidate. Finally, the Fieller/Fieller method which was used in [3], [4], and [5] appears to be dominated statistically for short runs by the jackknife method and more complex than the classical method which performs comparably for long runs.

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# DENUMERABLE STATE MARKOVIAN SEQUENTIAL CONTROL PROCESSES: ON RANDOMIZATIONS OF OPTIMAL POLICIES*

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**ABSTRACT**

We consider a denumerable state Markovian sequential control process. It is well known that when we consider the expected total discounted income as a criterion, there exists a nonrandomized stationary policy that is optimal. It is also well known that when we consider the expected average income as a criterion, an optimal nonrandomized stationary policy exists when a certain system of equations has a solution. The problem considered here is: if there exist two optimal nonrandomized stationary policies, will a randomization of these two policies be optimal? It is shown that in the discounted case the answer is always yes, but in the average income case, the answer is yes only under certain additional conditions.

## 1. INTRODUCTION

We consider a stochastic system which at times $n=0, 1, 2, \ldots$ can be classified to be in some state $i\epsilon\mathscr{L}$. We assume $\mathscr{L}$ to be a denumerable set. In each state $i\epsilon\mathscr{L}$, there are $K_i$ $(<\infty)$ available actions. There is a bounded reward $W(i, k)$ for taking action $k$ in state $i$. Let $X_n$ denote the state of the system at time $n$ and $\Delta_n$ denote the action taken at time $n$, $n=0, 1, 2, \ldots$. A basic assumption is that

$$P(X_{n+1}=j|X_0, \Delta_0, \ldots, X_n=i, \Delta_n=k)=p_{ij}^k \tag{1.1}$$

for all $i, j, k$ and $n$. We have

$$p_{ij}^k \geq 0 \qquad \text{for all } i, j, k, \tag{1.2}$$

$$\sum_j p_{ij}^k=1 \qquad \text{for all } i, k.$$

A policy $S$ for controlling the system is a set of functions $\{D_k(X_0, \Delta_0, \ldots, X_n)\}$ satisfying $0 \leq D_k(X_0, \Delta_0, \ldots, X_n) \leq 1$ for every $k$ and $\sum_k D_k(X_0, \Delta_0, \ldots, X_n)=1$ for every history $(X_0, \Delta_0, \ldots, X_n)$, $n=0, 1, 1, \ldots$. $D_k(X_0, \Delta_0, \ldots, X_n)$ is the probability of taking action $k$ at time $n$ when the particular history $X_0, \Delta_0, \ldots, X_n$ has occurred.

A policy $S$ is *Markovian if* $D_k(X_0, \Delta_0, \ldots, X_n)=D_{X_n,k}$, independent of $X_0, \Delta_0, \ldots, X_{n-1}$,

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$\Delta_{n-1}$. A policy $S$ is *nonrandomized Markovian* if $S$ is Markovian and $D_{X_n,k}=0$ or 1. A policy $S$ is *stationary* if $D_k(X_0, \Delta_0, \ldots, X_n=i)=D_{i,k}$. $S$ is *nonrandomized stationary* if $S$ is stationary and $D_{i,k}=0$ or 1. Note that for a stationary policy $S$, $\{X_n(S)\}$ is a Markov chain with stationary transition probabilities.

Observe that a nonrandomized Markovian policy $S$ is a sequence $\{f_n\}_{n=0}^{\infty}$, where each $f_n$ is an element of $F=\underset{i\in\mathscr{L}}{X}A_i$, $A_i$ being the set of available actions in state $i$. A nonrandomized stationary policy $S$ is then a sequence $\{f_n\}_{n=0}^{\infty}$ where each $f_n=f\in F$. We shall denote such a policy by $f^{(\infty)}$.

Two criteria for comparing different policies are the expected total discounted income and the expected average income. The first assumes a discount factor $\beta$, $0\leq\beta<1$, and the objective is to find a policy $S^*$ such that

$$I_\beta(S^*)=\sup_S I_\beta(S), \text{ where} \tag{1.3}$$

$$I_\beta(S)=E_S\left[\sum_{n=0}^{\infty}\beta^n W(X_n, \Delta_n)\right] \tag{1.4}$$

Under the expected average income criterion, the objective is to find a policy $S^*$ such that

$$I(S^*)=\sup_S I(S), \text{ where} \tag{1.5}$$

$$I(S)=\liminf_{N\to\infty} E_S\left[\frac{\sum_{n=0}^{N} W(X_n, \Delta_n)}{N+1}\right] \tag{1.6}$$

A policy $S^*$ satisfying (1.3) or (1.5) is termed *optimal*. (Note that $I_\beta(S)$ and $I(S)$ are column vectors, the $i$th element corresponding to the situation when $X_0=i$. Denoting the $i$th element of a vector $U$ by $U_i$, we will write $U\geq V$ if $U_i\geq V_i$ for all $i$, and $U>V$ if $U\geq V$ and $U\neq V$.)

Blackwell [1] and Derman [3] have shown that for the expected total discounted income criterion, there exists an optimal nonrandomized stationary policy. Derman [4] and Ross [7, 8] have shown that for the expected average income criterion, an optimal nonrandomized stationary policy exists if the system of equations

$$g+V(i)=\max_k\left\{W(i, k)+\sum_j p_{ij}^k V(j)\right\}, \qquad i\in\mathscr{L}$$

has a bounded solution.

When following a nonrandomized stationary policy $S$, at any time $n$, the current state of the system determines the action to be taken. There is no need for any record of the history of the system. Now, if there are two nonrandomized stationary policies, then in each state $i\in\mathscr{L}$, we have two possibly different actions to choose from. If in each state we choose randomly between the two actions, we intuitively expect the resulting policy to be optimal. Theorem 3.1 shows that this is true for the dis-

counted case, but example 4.1 shows that the same is not true for the average income case. However, Theorem 4.3 shows that such a result holds under certain conditions.

It may be observed that the ability to randomize between optimal policies also gives the controller some ability to evade hostile opposition.

## 2. SOME DEFINITIONS AND NOTATION

Suppose $f^{(\infty)}$ and $g^{(\infty)}$ are two nonrandomized stationary policies. Then, a policy $\Pi=(h_0, h_1, h_2, \ldots, h_n, \ldots)$ will be called a randomization of $f^{(\infty)}$ and $g^{(\infty)}$ if

$$P(h_n(i)=f(i))=\alpha_{n,i}$$

and

$$P(h_n(i)=g(i))=1-\alpha_{n,i},$$

where $0 \leq \alpha_{n,i} \leq 1$; $n=0, 1, 2, \ldots$; $i \epsilon \mathscr{L}$.

A policy $\Pi$ will be called a stationary randomization of $f^{(\infty)}$ and $g^{(\infty)}$ if it is a randomization of $f^{(\infty)}$ and $g^{(\infty)}$ and $\alpha_{n,i}=\alpha_i$ for all $n$.

The policy $(g_0, g_1, g_2, \ldots, g_m, f, f, \ldots, f \ldots)$ will be denoted by $(g_0, g_1, g_2, \ldots, g_m, f^{(\infty)})$. The column vector whose $i$th element is $W(i, f(i))$, $i \epsilon \mathscr{L}$, will be denoted by $W(f)$. The stochastic matrix whose $(i, j)$th element is $p_{ij}^{f(i)}$, $i, j \epsilon \mathscr{L}$, will be denoted by $P(f)$. If $S$ is the policy $\{f_n\}_{n=0}^{\infty}$, then $P_n[S]$ will denote the product $P[f_0] \cdot P[f_1] \cdot \ldots \cdot P[f_n]$.

## 3. THE DISCOUNTED CASE

If $S=\{f_n\}_{n=0}^{\infty}$, then we have

$$I_\beta(S)=W(f_0)+\sum_{n=1}^{\infty} \beta^n P_{n-1}(S) W(f_n)$$

$$=W(f_0)+\beta P(f_0) I_\beta(S'),$$

where $S'=\{f_n\}_{n=1}^{\infty}$.

THEOREM 3.1: If $\Pi=\{h_n\}_{n=0}^{\infty}$ is a randomization of two optimal policies $f^{(\infty)}$ and $g^{(\infty)}$, then $\Pi$ is optimal.

PROOF: Let

$$h_n(i)=f(i) \text{ with probability } \alpha_{n,i}$$

$$=g(i) \text{ with probability } 1-\alpha_{n,i},$$

$$n=0, 1, 2, \ldots; \qquad i \epsilon \mathscr{L}.$$

We first show that if $\Pi=(h_0, g^{(\infty)})$, *then* $\Pi$ is optimal. We have

$$I_\beta(\Pi) = W(h_0) + \beta P(h_0) I_\beta(g^{(\infty)}), \text{ where}$$

$$W(h_0) = \alpha \cdot W(f) + (1-\alpha) \cdot W(g);$$

$P(h_0)$ is the matrix whose $i$th row is

$$= \alpha \cdot i\text{th row of } P(f) + (1-\alpha) \cdot i\text{th row of } P(g);$$

$\alpha$ is the vector whose $i$th element is $\alpha_{0,i}$; $(1-\alpha)$ is the vector whose $i$th element is $(1-\alpha_{0,i})$; and $a \cdot b$ denotes the vector whose $i$th element is the product of the $i$th elements of the vectors $a$ and $b$. Thus,

$$\begin{aligned} I_\beta(\Pi) &= \alpha \cdot W(f) + (1-\alpha) \cdot W(g) + \beta[\alpha \cdot P(f) + (1-\alpha) \cdot P(g)] I_\beta(g^{(\infty)}) \\ &= \alpha \cdot [W(f) + \beta P(f) I_\beta(g^{(\infty)})] + (1-\alpha) \cdot [W(g) + \beta P(g) I_\beta(g^{(\infty)})] \\ &= \alpha \cdot I_\beta(f^{(\infty)}) + (1-\alpha) \cdot I_\beta(g^{(\infty)}) \\ &= I_\beta(g^{(\infty)}). \end{aligned}$$

Now assume $\Pi = (h_0, h_1, \ldots, h_m, g^{(\infty)}$ is optimal. Set $\Pi' = (h_0, h_1, \ldots, h_{m+1}, g^{(\infty)})$. Then

$$\begin{aligned} I_\beta(\Pi') &= W(h_0) + \sum_{n=1}^{m} \beta^n P_{n-1}(\Pi') W(h_n) + \sum_{n=m+1}^{\infty} \beta^n P_{n-1}(\Pi') W(h_n) \\ &= W(h_0) + \sum_{n=1}^{m} \beta^n P_{n-1}(\Pi) W(h_n) + \beta^{m+1} P_m(\Pi) [W(h_{m+1}) + \beta P(h_{m+1}) I_\beta(g)^{(\infty)}] \\ &= W(h_0) + \sum_{n=1}^{m} \beta^n P_{n-1}(\Pi) W(h_n) + \beta^{m+1} P_m(\Pi) I_\beta(h_{m+1}, g^{(\infty)}) \\ &= I_\beta(\Pi), \qquad \text{since } I_\beta(h_{m+1}, g^{(\infty)}) = I_\beta(g^{(\infty)}). \end{aligned}$$

Hence $\Pi'$ is optimal. Now let $\Pi = \{h_n\}_{n=0}^{\infty}$ and $(\Pi_n, g^{(\infty)}) = (h_0, h_1, \ldots, h_n, g^{(\infty)})$. Let $\epsilon > 0$ be given. Choose $n$ sufficiently large so that $\beta^n I_\beta(g^{(\infty)}) < \epsilon \cdot 1$, where 1 is the vector of 1's. We have

$$|I_\beta(\Pi) - I_\beta(g^{(\infty)})| \leq |I_\beta(\Pi) - I_\beta(\Pi_n, g^{(\infty)})| + |I_\beta(\Pi_n, g^{(\infty)}) - I_\beta(g^{(\infty)})| \leq \epsilon \cdot 1.$$

Hence the theorem.

## 4. THE AVERAGE INCOME CASE

It is interesting to observe that Theorem 3.1 is not true when we consider the average income criterion. Example 4.1 below illustrates this fact.

EXAMPLE 4.1: There are five states 1, 2, 3, 4, 5. There are two available actions, 1 and 2, in each of the states 1, 2 and 5, and only one action in states 3 and 4. We have $p^1_{12}=p^2_{13}=p^1_{23}=p^2_{21}=p^1_{34}=p^1_{45}=p^1_{53}=p^2_{51}=1$, $W(1,1)=W(1,2)=W(2,1)=W(2,2)=W(5,2)=0$, and $W(3,1)=W(4,1)=W(5,1)=1$.

Let $S_1$ be the policy that takes action 1 in all the states and $S_2$ be the policy that takes action 2 in the states 1 and 2 and action 1 in the other states. It can be verified that both $S_1$ and $S_2$ are optimal with

$$I(S_1)=I(S_2)=(1,1,1,1,1)'.$$

The policy $S$ that takes action 2 in state 2 and action 1 in all other states is a randomization of $S_1$ and $S_2$. However, $S$ is not optimal, since $I(S)=(0,0,1,1,1)'$.

However, Theorem 4.3 will show that, under certain conditions, a stationary randomization of two optimal nonrandomized stationary policies is optimal.

THEOREM 4.2: If $S^*$ is an optimal nonrandomized stationary policy such that $\{X_n(S^*)\}$ is positive recurrent over $A_{S*}\subset\mathscr{L}$ and if $m_{ij}$, the mean first passage time from state $i$ to state $j$ under $S^*$, is finite for all $i\epsilon A'_{S*}$, $j\epsilon A_{S*}$, then the equations:

$$g+V(i)=\max_k\left\{W(i,k)+\sum_j p^k_{ij}V(j)\right\},\qquad i\epsilon A_{S*} \tag{4.1}$$

have a solution in $g$ and $V(i)$ and $k_i$, the action prescribed by $S^*$ in state $i$, satisfies (4.1), $i\epsilon A_{S*}$. ($A'_{s*}$ is the complement of $A_{s*}$)

PROOF: Since $m_{ij}<\infty$ for all $i\epsilon\mathscr{L}$, $j\epsilon A_{S*}$, from Theorem 1 of Derman and Veinott [5], it follows that the equations

$$g+V(i)=W(i,k_i)+\sum_j p^{k_i}_{ij}V(j),\qquad i\epsilon\mathscr{L} \tag{4.2}$$

have a solution in $g$ and $V(i)$, $i\epsilon\mathscr{L}$. Assume that for some state, say state 0, in $A_{S*}$, we have

$$g+V(0)=W(0,k_0)+\sum_j p^{k_0}_{0j}V(j)<\max_k\left\{W(0,k)+\sum_j p^k_{0j}V(j)\right\}. \tag{4.3}$$

If $c_{i0}$ denotes the expected income during a first passage from state $i$ to state 0 under $S^*$, from Theorem 1 of Derman and Veinott [5], we see that

$$g=c_{00}/m_{00},\qquad V(i)=c_{i0}-gm_{i0},\qquad i\epsilon\mathscr{L}$$

is a solution of (4.2). If $I(i,S^*)$ is the $i$th element of $I(S^*)$, note that $I(i,S^*)=g$ for all $i\epsilon\mathscr{L}$ and $V(0)=0$. Let $k'_0$ be such that

$$W(0,k'_0)+\sum_j p^{k'_0}_{0j}V(j)=\max_k\left\{W(0,k)+\sum_j p^k_{0j}V(j)\right\}. \tag{4.4}$$

Let $S$ be a policy that is identical with $S^*$ in all states except the state 0, and in state 0, $S$ prescribes the action $k_0'$. Since under the policy $S$, the state 0 is a positive recurrent state, we have

$$I(0, S) = \frac{W(0, k_0') + \sum_{j \neq 0} p_{0j}^{k_0'} C_{j0}}{1 + \sum_{j \neq 0} p_{0j}^{k_0'} m_{j0}}. \tag{4.5}$$

Since $V(0) = 0$, we also have

$$g < W(0, k_0') + \sum_{j \neq 0} p_{0j}^{k_0'} V(j) = W(0, k_0') + \sum_{j \neq 0} p_{0j}^{k_0'} (C_{j0} - g m_{j0}) \tag{4.6}$$

that is,

$$g < \frac{W(0, k_0') + \sum_{j \neq 0} p_{0j}^{k_0'} C_{j0}}{1 + \sum_{j \neq 0} p_{0j}^{k_0'} m_{j0}} = I(0, S), \tag{4.7}$$

which contradicts the optimality of $S^*$. Hence the theorem.

THEOREM 4.3: Let $S_1$ and $S_2$ be two optimal nonrandomized stationary policies and let $S$ be a stationary randomization of $S_1$ and $S_2$. If

(a) $\{X_n(S_1)\}$, $\{X_n(S_2)\}$, and $\{X_n(S)\}$ are positive recurrent over $A \subset \mathscr{L}$, and
(b) for some state 0 in $A$, the set $\{m_{i0}, i \in \mathscr{L}\}$ is bounded under $S_1$, $S_2$, and $S$,

then $S$ is optimal.

PROOF: Since, with probability one, $X_n(S)$ will be in $A$ for finite $n$, without loss of generality, we may assume $X_0$ to be in the class $A$.

Let $k_i$ and $k_i'$ be the actions prescribed by $S_1$ and $S_2$, respectively, in state $i$, $i \in \mathscr{L}$. Let $S$ be a policy that prescribes, in state $i$, action $k_i$ with probability $\alpha_i$ and action $k_i'$ with probability $(1 - \alpha_i)$, $0 \leq \alpha_i \leq 1$, $i \in \mathscr{L}$. From Theorem (4.2) we have

$$W(i, k_i) + \sum_j p_{ij}^{k_i} V(j) = W(i, k_i') + \sum_j p_{ij}^{k_i'} V(j) = \max_k \left\{ W(i, k) + \sum_j p_{ij}^k V(j) \right\} = g + V(i) \qquad i \in A.$$

Hence,

$$\begin{aligned} g + V(i) &= \max_k \left\{ W(i, k) + \sum_j p_{ij}^k V(j) \right\} \\ &= \alpha_i W(i, k_i) + (1 - \alpha_i) W(i, k_i') + \sum_j \{ \alpha_i p_{ij}^{k_i} + (1 - \alpha_i) p_{ij}^{k_i'} \} V(j), \qquad i \in A. \end{aligned}$$

As in Theorem 1.2 of Ross [7], we have, for all $t$,

$$E_S\left\{\sum_{n=1}^{t}\left[V(X_n)-E_S(V(X_n)\mid H_{n-1})\right]\right\}=0, \tag{4.8}$$

where $H_{n-1}=(X_0, \Delta_0, \ldots, X_{n-1}, \Delta_{n-1})$ = history up to time $n$. Since

$$E_S[V(X_n)\mid X_{n-1}=i]=\sum_j\left\{\alpha_i p_{ij}^{k_i}+(1-\alpha_i)p_{ij}^{k_i'}\right\}V(j)$$

$$=g+V(i)-\{\alpha_i W(i, k_i)+(1-\alpha_i)W(i, k_i')\},$$

from (4.8) we get

$$g=I(X_0, S)+\lim_{t\to\infty}E_S\frac{V(X_t)}{t}. \tag{4.9}$$

Since, by assumption, $\{m_{i0}, i\in\mathscr{L}\}$ is bounded under $S_1$, $S_2$, and $S$, it follows that $\{V(i), i\in\mathscr{L}\}$ is also bounded. (In Theorem 4.2, it has been observed that $V(i)=c_{i0}-gm_{i0}$). Hence it follows that

$$\lim_{t\to\infty}E_S\frac{V(X_t)}{t}=0.$$

Thus, $I(X_0, S)=g=I(X_0, S_1)=I(X_0, S_2)$, which completes the proof.

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# STOCHASTIC DUELS WITH ROUND-DEPENDENT HIT PROBABILITIES

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**ABSTRACT**

The effect of round dependent hit probabilities in the fundamental stochastic duel are examined. The general solution and several specific examples are derived where one side's hit probabilities are improved from round to round. For these specific cases the advantages of round to round improvement are explicitly displayed.

## INTRODUCTION

Most of the literature on stochastic duels has dealt with situations where the two contestants have fixed (not necessarily equal) hit probabilities on each round fired. Here, we are concerned with the possibility that one or both contestants may have changing hit probabilities as successive rounds are fired. In particular, we wish to examine the improvement in the probability of winning which occurs if the hit probability is a monotone nondecreasing function of round number. This we would expect to be the case where the effect of fired rounds can be observed and sensed as in many armored combat situations or in observed-fire artillery exchanges.

We shall limit our discussion to the case where the duel terminates when one of the contestants is hit and where the time between rounds fired is a continuous random variable. We also start with unloaded weapons and have unlimited ammunition and unlimited time. Thus, we are focussing our attention entirely on the effect of round-dependent hit probabilities.

First, consider a single marksman firing at a passive target. If his unconditional hit probability is given by $p_n$ for the $n$th round (and may vary from round to round), then the probability that he scores his *first* hit (which terminates his firing) on the $n$th round is

$$\rho_n = p_n \prod_{j=1}^{n-1} (1-p_j) = p_n \prod_{j=1}^{n-1} q_j,$$

where $q_j = 1 - p_j$.

There have been several previous studies of certain aspects of this model. Williams [9] took $\rho_n$ in Equation (1) to be a negative binomial. This allowed two parameters for flexibility and assured that $p_n$ was increasing. He also assumed the times between firings were distributed as negative exponentials. Finally, he took as the criterion of comparison $p = \frac{1}{\bar{n}}$, where $\bar{n} = \sum_{i=1}^{\infty} i\rho_i$. This meant that $p_1 \leq p \leq p_\infty$, which does not permit a comparison based on $p_1$, the first round hit probability. Our assumption is that the first round hit probability for a marksman with changing hit probability is the same as the hit

probability the same marksman would have if he had no change from round to round (i.e. if he had a fixed hit probability).

Later, Bhashyam and Singh [5] obtained a general solution for negative exponential firing times only with limited ammunition supplies and with $p_n$ variable. They considered both increasing and decreasing $p_n$'s and used $p_1$ as the comparison criterion. Their examples do not explicitly show the improvement that sensing can give. Still later, using the supplementary variable and transform techniques, Bhashyam [6] obtained the general solution for ammunition-limited, round-dependent duels with general firing time distributions. Although he specialized these to negative exponential and general Gamma firing time distributions, he gave no examples for particular varying $p_n$'s.

Finally, it should be noted that in papers by Barfoot [2], [3], [4], Finley [7] and Kimbleton [8] Markov-dependent hit probabilities are introduced. This concept does not contain the idea of changing hit probabilities dealt with here, but does introduce the notion of a state space of size greater than 2 (hit, nonhit).

## THE MODEL

Proceeding in our usual manner [1], [10] and using Equation (1), we may at once write down the characteristic function $(\Phi(u))$ of the marksman's time-to-a-hit density function $(h(t))$ as

$$\Phi(u)=p_1\phi(u)+p_2(1-p_1)\phi^2(u)+p_3(1-p_1)\,(1-p_2)\phi^3(u)+\ldots + p_n(1-p_1)\,(1-p_2)\ldots(1-p_{n-1})\phi^n(u)+\ldots \tag{2}$$

$$=\sum_{n=1}^{\infty}\phi^n(u)p_n\prod_{j=1}^{n-1}(1-p_j)=\sum_{n=1}^{\infty}\phi^n(u)p_n\prod_{j=1}^{n-1}q_j=\sum_{n=1}^{\infty}\rho_n\phi^n(u)$$

where $\phi(u)$ is the characteristic function of the marksman's interfiring time density function, $f(t)$.

By rearranging terms, Equation (2) may be rewritten as

$$\Phi(u)=\phi(u)-[1-\phi(u)]\sum_{n=1}^{\infty}\phi^n(u)\prod_{j=1}^{n}q_j. \tag{3}$$

If we now turn to the duel between two contestants ($A$ and $B$), then $A$'s chances of winning [10] are given by

$$P[A]=\frac{1}{2\pi i}\int_L \Phi_A(-u)\Phi_B(u)\,\frac{du}{u}, \tag{4}$$

where $\int_L$ refers to contour integration in the complex plane around the contour shown in Figure 1., where $\rho$ is finite but less than the distance to the nearest singularity and the integral must vanish on $C$ as $R\to\infty$. Also,

$$P[B]=1-P[A] \tag{5}$$

as there is zero probability of a draw.

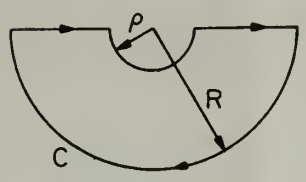


FIGURE 1. Path of integration for Equation (4).

## SPECIFIC CASES

1. First, let us use a monotone, decreasing miss probability given by

$$q_j = \left(\frac{N}{j} - 1\right) a = \frac{(N-j)}{j}\, a; \qquad j = 1, 2, \ldots, N. \tag{6}$$

$$= 0; \qquad j = N+1, N+2, \ldots .$$

This expression has two parameters to allow for flexibility in adjusting to real situations namely, N and $a$. It decreases monotonely to zero on the $N$th round (when a hit is a certainty).
We also note that

$$q_1 = (N-1)\, a \leqslant 1, \tag{7}$$

which limits the relationships between $a$ and $N$ to

$$a \leqslant \frac{1}{N-1}; \qquad \begin{matrix} N \epsilon I^+ \\ a > 0, \end{matrix} \tag{8}$$

which keeps $0 \leqslant q_1 \leqslant 1$, as it should.

If we now substitute Equation (6) into (3), after some manipulation, we have

$$\Phi(u) = \phi(u) - [1 - \phi(u)] \sum_{n=1}^{N-1} [a\phi(u)]^n \binom{N-1}{n} \tag{9}$$

$$= 1 - [1 - \phi(u)]\, [1 + a\phi(u)]^{N-1}.$$

Now, let $A$ and $B$'s firing-times be negative exponential with rates $r_A$ and $r_B$, respectively, and thus

$$\phi_A(-u) = \frac{r_A}{r_A + iu} \text{ and } \phi_B(u) = \frac{r_B}{r_B - iu}. \tag{10}$$

If we let $A$ have an increasing hit probability according to Equation (6), where we call his first-round miss probability from (7) $q_A$ and if we let $B$ have a *constant* miss probability $q_B$ ($p_B = 1 - q_B$) then from Equation (9)

(11) $$\Phi_A(-u) = 1 - \frac{iu}{(r_A + iu)^N}[(1+a)r_A + iu]^{N-1},$$

and from Reference [10]

(12) $$\Phi_B(u) = \frac{p_B r_B}{p_B r_B - iu}.$$

Equations (11) and (12) are substituted into (4) to give

(13) $$P[A] = \frac{1}{2\pi i}\int_L \left\{1 - \frac{iu}{(r_A+iu)^N}[(1+a)r_A+iu]^{N-1}\right\}\left(\frac{p_B r_B}{p_B r_B - iu}\right)\frac{du}{u}.$$

The integrand of (13) has one simple pole in the lower half-plane at $u=-p_B r_B i$, which, by using the residue theorem, gives

(14) $$P[A] = 1 - \frac{1}{1+x}\left[1 + \left(\frac{q_A}{N-1}\right)\left(\frac{x}{1+x}\right)\right]^{N-1}$$

where

$$x = \frac{r_A}{p_B r_B} \quad \text{and} \quad a = \frac{q_A}{N-1}.$$

2. Now, instead of letting the improving firer achieve a hit with certainty on the $N$th round, let us leave him with a residual error no matter how many rounds he fires. So suppose instead of (6), we use

(15) $$q_j = \frac{q_1}{j}, \qquad j = 1, 2, \ldots$$

where $q_1$ is his first round miss probability. This can be achieved by taking Lim $q_j$ in equation (6),

$$\begin{matrix} N \to \infty \\ a \to 0 \end{matrix} \ni (N-1)\, a = q_1,$$

and taking the same limit in Equation (9) to obtain

(16) $$\Phi(u) = 1 - [1 - \phi(u)] \exp [q_1 \phi(u).$$

Of course, Equation (16) may be obtained directly from (3) by substituting (15) and closing the expression. Now, to show the versatility of this technique, let $A$ be round-improving according to Equation (15), where we shall replace $q_1$ by $q_A$, and also let his inter-firing time be $\gamma(2)$ with characteristic function

$$\phi_A(-u)=\left(\frac{r_A}{r_A+iu}\right)^2. \tag{17}$$

At the same time, let $B$ have constant hit probability $p_B$ and negative exponential firing time with rate $r_B$. Hence,

$$\Phi_A(-u)=\frac{(r_A+iu)^2+u(u-2ir_A)\exp\left[\dfrac{q_A r_A^2}{(r_A+iu)^2}\right]}{(r_A+iu)^2} \tag{18}$$

and $\Phi_B(u)$ is given by (12).

Substituting these expressions into Equation (4) and evaluating as before, we have

$$P[A]=\frac{(1+x)^2-(1+2x)\exp\left[q_A\left(\dfrac{x}{1+x}\right)^2\right]}{(1+x)^2}, \tag{19}$$

where

$$x=\frac{r_A}{p_B r_B}.$$

3. Finally, to get expressions which are easier to use to assess the importance of improving hit-probability let $A$'s marksmanship improve as in Example 2 above, but let him have negative exponential inter-firing times with rate $r_A$. $B$ remains the same as in Example 2 above. Then by using the appropriate substitutions

$$P[A]=\frac{1}{2\pi i}\int_L \frac{r_A+iu\left[1-\exp\left(\dfrac{q_A r_A}{r_A+iu}\right)\right]}{r_A+iu}\left(\frac{p_B r_B}{p_B r_B-iu}\right)\frac{du}{u}, \tag{20}$$

which evaluates as before to

$$P[A]=\frac{x+1-\exp\left[q_A\left(\dfrac{x}{1+x}\right)\right]}{1+x} \tag{21}$$

where

$$x=\frac{r_A}{p_B r_B}.$$

Equation (21) may also be obtained directly from Equation (14) by taking the limit as $N \to \infty$. Equation (21) is plotted as contour maps in the $p_A$, $p_B$ plane in Figure 2. When we compare these plots with the equivalent ones (Figure 2) in Reference [10], where the duel is exactly the same except that $A$ has constant hit probability; we see that they have a vastly different appearance. In the latter, all contours radiate from the origin whereas in the duel considered here, none do. On examination, it is apparent that $A$'s chances of winning have greatly improved, but exactly how much is not apparent. In passing, we note that in Figures 2(b) and 2(c) some contours are missing. This is so because in this duel, $P[A]$

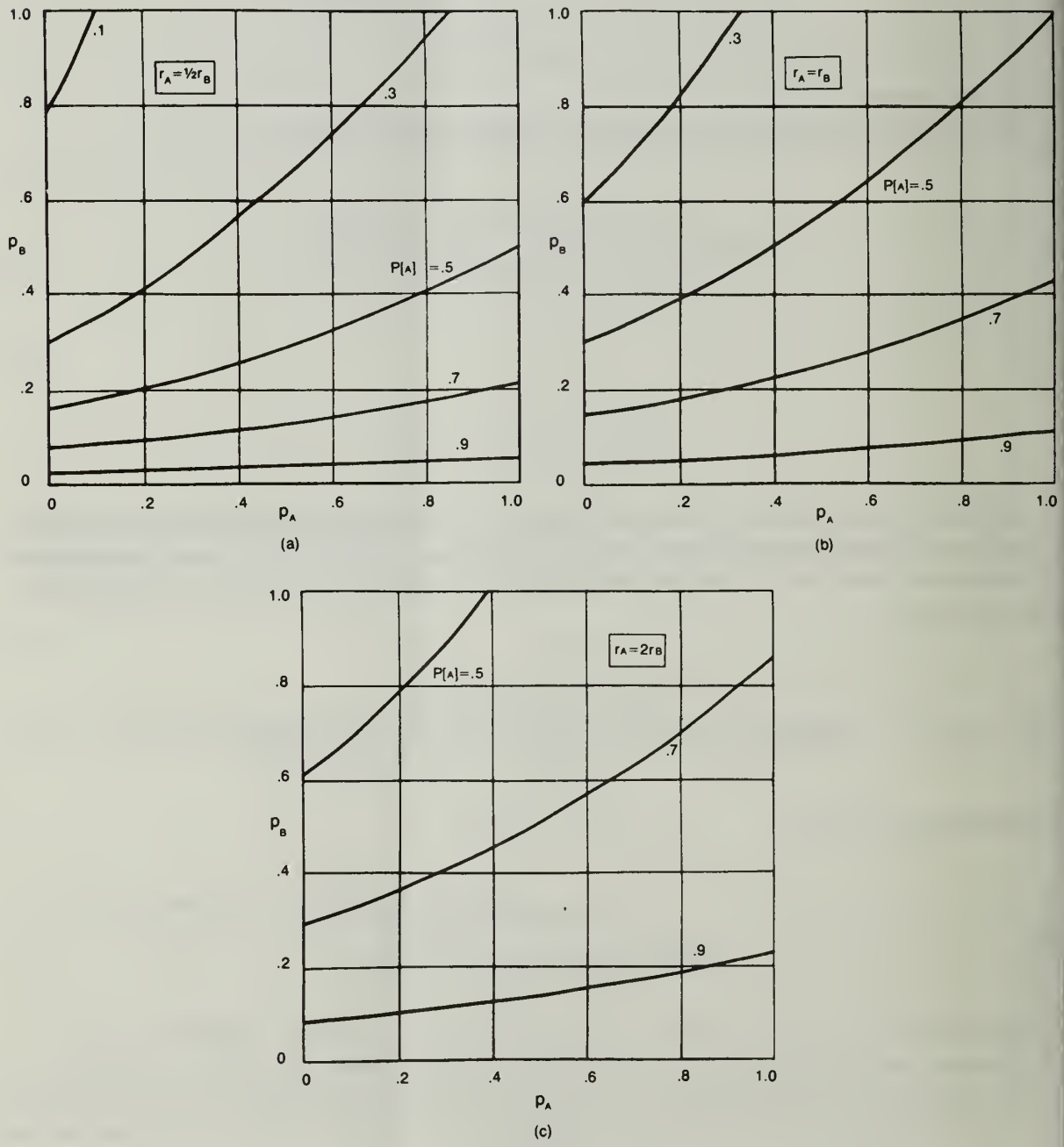


FIGURE 2. Contour maps of Equation (21).

has a lower bound different from zero. In fact, if we let $p_B=1$ and $p_A=0$, we get

$$P[A]_{\min}=1-\left(\frac{r_B}{r_A+r_B}\right)\times\exp\left[\frac{r_A}{r_A+r_B}\right],$$

which for Figures 2(a), (b), (c) are 0.0695, 0.1756 and 0.351, respectively.

If we now compare this duel with the fundamental duel with negative exponential firing times (i.e. the only difference is that $A$ can improve his marksmanship from round to round), we can get a measure of the help that this improvement may be. From [10] we have that $A$'s winning probability for the fundamental duel is

$$P[A]_f=\frac{p_Ar_A}{p_Ar_A+p_Br_B}. \tag{22}$$

The measure we seek is thus $\dfrac{P[A]}{P[A]_f}$, where $P[A]$ is from Equation (21) and $P[A]_f$ from (22). And,

$$\frac{P[A]}{P[A]_f}=\frac{\left(1+\dfrac{1}{p_Ax}\right)\left\{x+1-\exp\left[\dfrac{x(1-p_A)}{1+x}\right]\right\}}{1+x}, \tag{23}$$

where

$$x=\frac{r_A}{p_Br_B}.$$

This result is plotted in Figure 3. This figure clearly indicates that for small $p_A$, $A$'s chances are substantially improved, but as $p_A$ increases his advantage lessens. It is also interesting to note that for fixed $p_Br_B$, as $r_A$ increases $P[A]/P[A]_f$ increases to a maximum and then decreases slowly to an asymptote of 1 at $\infty$. The maximum is near the point where $A$'s firing rate ($r_A$) is equal to $B$'s hit rate ($p_Br_B$).

## SUMMARY

In this paper, we have examined the effect of round dependent hit probabilities in the fundamental stochastic duel. The general solution and several specific examples have been derived where one side's hit probabilities are improved from round to round. For these specific cases, the advantages of round to round improvement are explicity displayed. The case where a contestant's performance deteriorates from round to round can also be determined from the general case.

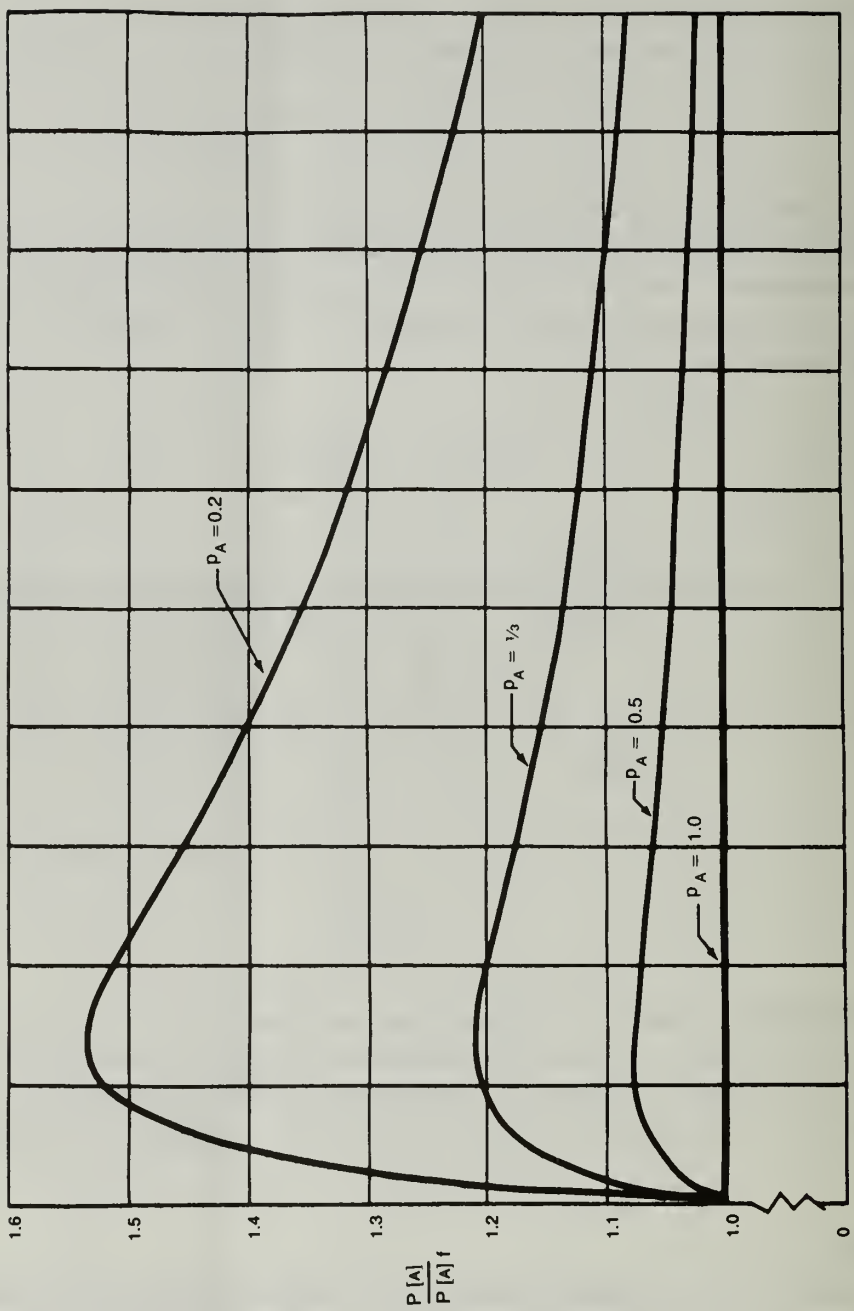


FIGURE 3. Plot of Equations (23).

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# ONE MACHINE SEQUENCING TO MINIMIZE MEAN FLOW TIME WITH MINIMUM NUMBER TARDY

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**ABSTRACT**

The problem of sequencing $n$ jobs on one machine is considered, under the multiple objective of minimizing mean flow time with the minimum number of tardy jobs. A simple procedure is first proposed to schedule for minimum flow time with a specified subset of jobs on time. This is used in conjunction with Moore's Algorithm in a simple heuristic producing good and often optimal schedules. A branch-bound algorithm is presented to produce the optimal schedule efficiently with the help of several theorems which eliminate much branching.

## 1. INTRODUCTION

The problem of optimally scheduling a set $N=\{1, 2, \ldots, n\}$ of $n$ jobs on one machine (the $n/1$ problem) has been considered for a variety of criteria [2]. We shall write $n/1/C$ for such a problem, where $C$ is the criterion to be minimized. Associated with job $i$ is its processing time $p_i$ and its due date $d_i$. All jobs are available for processing at time $t=0$. A schedule or sequence $S$ produces a completion or flow time $F_i=F_i(S)$ for each job, and consequently a tardiness $T_i=\max(0, F_i-d_i)$.

Probably the two most fundamental results in this context are:

(1) To minimize the mean flow time, $\bar{F}$, of all jobs, or equivalently to minimize the total flow time $F=\sum_{i=1}^{n} F_i$, jobs should be sequenced in order of nondecreasing processing time, the shortest processing time (SPT) schedule.

(2) To minimize the maximum lateness or maximum tardiness ($T_{\max}$), jobs should be sequenced in order of nondecreasing due dates, the earliest due date (EDD) schedule.

For the objective of minimizing the number of tardy jobs (the $n/1/\#T$ problem), a simple algorithm due to Moore [6] with refinement by Hodgson [6] and Emmons [3] is implicit in the following theorem. We incorporate the generalization due to Sidney [8], in which a subset, $E$, of jobs is given which must be done on time (i.e., we require $T_i=0$, $i\epsilon E$). Assume jobs are numbered in nondecreasing order of due dates, so that EDD $=(1, 2, \ldots, n)$.

THEOREM 1.1 (Moore): For the $n/1/\#\mathrm{T}|\mathrm{E}$ problem, schedule job $k$ last,* where:

(1) Job $j$ is the first tardy job in EDD;

(2) $J=\{1, 2, \ldots, j\}$;

(3) $p_k=\max_{i\epsilon J\cap E'} p_i$;

and $J\cap E'$ is the set of jobs in $J$ that are not in $E$ (we denote the complement of a set by a prime:

*More precisely, there exists an optimal schedule in which job $k$ is last.

$E' = N - E$). Here and hereafter, unless otherwise specified, we assume that if more than one job $k$ satisfies a criterion, ties are broken arbitrarily.

Moore's Algorithm (MA) singles out one job, $k$, after another which may be put into a set, $L$, of late jobs ($T_k > 0$) and guarantees that at the end the cardinality of $L$, which we denote $|L|$, is minimal. Given only the $\#T$ objective, those which are selected to be tardy are simply placed at the end without further consideration; the amount of their tardiness is of no concern. It is clear that with further thought the schedule produced by MA could be improved with regard to a secondary objective, without increasing the $\#T$. Not only could the jobs be rearranged while maintaining the same set $L$ of jobs tardy, but, in general, many other sets of jobs with equal cardinality could be chosen to be tardy.

It may be that a manager's chief concern is customer satisfaction, which he chooses to measure in terms of the number of jobs done on time. As a secondary measure, he may wish to consider shop efficiency combined with general speed of service to all customers, and hence he may choose total flow time, $F$, so as to minimize average delivery time, in-process inventory, etc., (see [2], p. 15). It is this compound objective, $F|\min \#T$, that we shall consider.

A special case of this problem has been considered by Smith [9], on the assumption that no job need be tardy (this can easily be determined by checking whether EDD gives $T_{\max} = 0$). His result:

THEOREM 1.2 (Smith): For the $n/1/F|\#T = 0$ problem, schedule job $k$ last, where

$$p_k = \max_{i \in M} p_i,$$

and

$$M = \{i : d_i \geq \Sigma_{i \in N} p_i\}$$

is the set of jobs which are not tardy even when processed last.

This result has been extended by Heck and Roberts [5] to the $n/1/F|\min t_{\max}$ problem, and further extended by Emmons [4] to the case $n/1/F|\min c_{\max}$, where for any schedule $c_{\max} = \max_i c_i(F_i)$ and $c_i(t)$ is an arbitrary nondecreasing function representing the cost of completing job $i$ at time $t$. The present paper discusses an extension of Smith's result in another direction.

## 2. MINIMUM FLOW TIME WITH PARTIAL DUE DATE CONSTRAINTS

We shall start by establishing a preliminary result which, besides being of interest in itself, will be needed in the later development. Suppose a subset, $E$, of high priority jobs is given, each of which must be done by its due date. Define a *feasible* schedule as one in which no job $i \in E$ has $T_i > 0$. For all other jobs, due dates are not given, or if given will not be considered. We wish to minimize the mean (or total) flow time, subject to the above constraint. A simple algorithm follows in the obvious way from the following theorem.

THEOREM 2.1: For the $n/1/F|E$ problem, schedule job $k$ last, where

$$p_k = \max_{i \in M \cup E'} p_i$$

and $M \cup E'$ is the set of jobs which either may be tardy or are not tardy even when last.

PROOF: Suppose that in some feasible schedule the rule is violated, so that job $j$ is last, with $p_j < p_k$. Interchanging jobs $j$ and $k$ diminishes $F$, and the schedule remains feasible since $k \in M \cup E'$.

Q.E.D.

## 3. MINIMUM FLOW TIME WITH MINIMUM NUMBER TARDY

Now, for the objective $F|\min \#T$, the above results can be applied once we find the set, $L$, of minimum cardinality that we wish to make tardy: we simple define $E = N - L$ and apply Theorem 2.1. There are generally several such sets, and we seek the one which produces the minimal $F$.

An obvious candidate for $L$ is the set of jobs produced by MA; we must run through that algorithm first, anyway, to determine the min $\#T$. This set of jobs would appear to have just the right properties, since MA always recommends as the job to join $L$ the largest of a set of jobs, one of which must be tardy. This is in perfect accord with our desire to put longer jobs later, to achieve the minimum $F$. Surprisingly, this is not always the optimal set. For example, if three jobs (1, 2, 3) have processing times (3, 1, 4) and due dates (3, 4, 5), respectively, then MA tells us that min $\#T = 1$, and selects job 3, the longest job, to be tardy. With $L = \{3\}$ and hence $E = \{1, 2\}$, Theorem 2.1 gives the schedule (1, 2, 3) with $F = 15$. However, the schedule which gives the minimal $F|\min \#T$ is (2, 3, 1), with $L = \{1\}$, $F = 14$. This hints at the complexity of the problem: the changing of one job in $L$ may permit (or force) a complete reordering of the jobs in $N - L$.

One thing remains true: at each decision point of MA, a set of jobs is identified (the set $J$ of Theorem 1.1) such that at least one job in that set must be late. This suggests a branch-bound algorithm, with each decision point or node of the decision tree corresponding to a decision point of MA. Each branch represents the choice of one of the jobs in the set $J$ for inclusion in $L$. To be more precise, we shall hereafter mean by $L$ the complete *optimal* set of late jobs. At any node, $a$, the partial set of jobs that have been selected to be tardy up to that point will be the *late set*, $L_a$. $L_a$ is generally smaller than $L$, and at some nodes will not even be a subset of $L$.

We next prove some theorems that eliminate many of the branches at each node and enable us to compute bounds.

THEOREM 3.1: Suppose job $j$ is not tardy in the SPT schedule. Then it need never be considered for inclusion in $L$.

PROOF: Suppose $j$ satisfies the hypothesis, and $j \in L$ for an optimal schedule $S$. Then it must start later in $S$ than in SPT. Thus, there is at least one job $k$ that follows $j$ in SPT (i.e., $p_k \geq p_j$) which has the properties:

(1) $k$ precedes $j$ in $S$;

(2) $k$ starts no later in $S$ than $j$ starts in SPT.

But now, if we interchange $j$ and $k$ in $S$, $F$ can only be diminished, while the number tardy has not increased, since by property (2) we now have $j$ on time and only $k$ can have been made tardy. We now have a schedule at least as good as $S$ with $j \notin L$. Q.E.D.

We define for each node $a$ (numbered in the order they are generated) an *early set*, $E_a$, of jobs that will not be considered for inclusion in $L$ either here or at subsequent nodes: jobs that we are sure will be early in an optimal schedule. This set may change from node to node, increasing as we work down the tree. We can now initialize $E_1$ with all jobs not tardy in SPT. Also, include all jobs that follow the last decision point of MA. The set $E_a$ may be used in Theorem 2.1 to compute the minimal $F$ given that no $i \in E_a$ is tardy. While the schedule so produced may be infeasible for the present problem (too many tardy jobs), it will give us a lower bound which can be used for node $a$. Of course, the larger $|E_a|$, the better the bound will be. We next observe how to augment successive $E_a$.

From any node with several branches, the possibility exists of generating the same late set several

times. For example, if node $a$ branches to nodes $a+1$ and $a+2$ which assign jobs $j_1$ and $j_2$ to $L_a$, respectively, a later branch below $a+1$ may assign $j_2$, and vice versa. To block the double generation of any late set including both $j_1$ and $j_2$, we can add $j_1$ to the set $E_{a+2}$ or $j_2$ to the set $E_{a+1}$. The latter will be more convenient for our purposes. In general:

LEMMA 3.2: If node $a$ with $E_a$ and $L_a$ branches to $b$ new nodes $a+1, \ldots, a+b$ which assign jobs $j_1, \ldots, j_b$ to $L_a$, respectively, then without loss of optimality we can set $E_{a+b}=E_a, E_{a+b-1}=E_a \cup \{j_b\}$, and generally $E_{a+i}=E_{a+i+1} \cup \{j_{i+1}\}$ for $i=1, \ldots, b-1$.

Since the new nodes can be order arbitrarily, it will be convenient to let $p_{j_1} \leqslant p_{j_2} \leqslant \ldots$, so that the largest numbered node will have the largest processing time associated with its job, and the smallest set, $E_a$. Note that the lower bounds (LB) obtained for successive nodes $a+i$ $(1 \leqslant i \leqslant b)$ form a monotone sequence, since the sets $E_{a+i}$ are nested. Thus, if one node, say $a+j$, has LB large enough to reject it, all other nodes $a+i$, $0<i<j$, can be simultaneously rejected.

Finally, we present one more concept which can be used to further diminish the number of branches from any node. Define $A_a$ to be the set of jobs at node $a$ which are available to be added to $L_a$. By the operation of MA, this includes all the jobs in EDD (omitting those in $L_a$) up to and including the job whose tardiness precipitated the decision, except those in $E_a$. For jobs $i$ and $j$ in $A_a$ at node $a$, we shall say that $i$ dominates $j$ ($iDj$) if it is better to add $i$ to $L_a$ than to add $j$, in the sense that $\min \bar{F}|L_{i,j'} < \min \bar{F}|L_{j,i'}$ where

$$L_{i,j'} = \text{any minimal set of tardy jobs including } i, \text{ but not } j.$$

Also, $iDj$ if no set $L_{j,i'}$ exists.

THEOREM 3.3: At any node, only undominated jobs need be considered for addition to $L$.

PROOF: Suppose $i_d Dj$ (there may be more than one job $i_d \in A_a$ with this property), with $d$ decisions, including this one, remaining. If we are to end up with $j \in L$ we must have $i_d \in L$ also; since if only one of them is to be in $L$, it must be $i_d$.

If $j$ as well as $i_d$ belongs in $L$, $j$ must become undominated at one of the remaining $n-1$ decisions; for if not, there is a set of jobs $i_d, i_{d-1}, \ldots, i_1$ each of which dominates $j$ at one of the $d$ decisions, so each of them must be included in $L$ in preference to $j$, leaving no room for $j$. Q.E.D.

Note that the number $d$ is always known, and invariant with the jobs selected. This is because MA (which has been run through as a preliminary step) tells us $|L| = \min \#T$, and as each decision adds one job to $L$, we always know how many decisions remain. Of course, the wrong choice at some point could lead us down a path where a larger number must be tardy, but such branches are abandoned as soon as this becomes clear.

THEOREM 3.4: Suppose at node $a$ there exist jobs $i$ and $j$ in $A_a$ such that

(1) $p_i \geqslant p_j$;

(2) $p_i - p_j \geqslant d_i - d_j$.

Then $iDj$.

PROOF: In any schedule $S$ giving $\min F|L_{j,i'}$, since $j$ is tardy and $i$ is not tardy by definition of $L$, we have using (2):

$$F_i \leqslant d_i \leqslant d_j + p_i - p_j < F_j + p_i - p_j.$$

But if $i$ follows $j$ in $S$, $F_i \geq F_j + p_i$. Therefore $i$ precedes $j$ in $S$. Now interchange $i$ and $j$ to get a new schedule $S'$, in which

$$F_j' = F_i - p_i + p_j \leq F_i - d_i + d_j \leq d_j,$$

so that $j$ is on time in $S'$. Since any jobs between $i$ and $j$ can only be advanced by the interchange, the $\#T$ cannot have increased and $F$ can only have decreased. Thus, we have improved the schedule by making $i$ tardy instead of $j$. Q.E.D.

It may be worth pointing out the simpler though weaker version:

COROLLARY 3.5: $iDj$ if $p_i \geq p_j$ and $d_i \leq d_j$.

Of course, whenever $iDj$, job $j$ can be removed from the set $A_a$.

## 4. THE ALGORITHM

We shall speak of *expanding* a node to mean producing all necessary branches from that node. As each new node is produced, it is entered on the *node list*. A node, $a$, is defined by the sets of jobs $E_a$ and $L_a$. A branch from $a$ corresponds to the assignment of an additional job to $L_a$. Assume jobs are numbered in EDD sequence.

1. Enter node $a = 1$ on the node list, with $L_1$ empty and $E_1$ the set of all jobs which are not tardy in the SPT schedule (with later due dates first, in case of equal processing times).
2. Choose as the next node, $a$, for expansion the largest numbered node remaining on the node list; and remove it from the list.
3. With jobs in $L'_a$ arranged in EDD order, let $j$ be the first tardy job and $J_a = \{1, \ldots, j\} \cap L'_a$. If no job is tardy, go to step 7.
4. Check the jobs in $J_a$ for dominance. Let $B_a$ be the set of undominated jobs in $J_a \cap E'_a$, with $|B_a| = b$.
5. Add to the node list a new node for each job $j_i \in B_a$, $i = 1, \ldots, b$, numbering them $a+1$, $a+2, \ldots, a+b$ in increasing order of $p_{j_i}$, with $L_{a+i} = L_a \cup \{j_i\}$ and $E_{a+i} = E_a \cup C_i$, where $C_i$ is the set of the $b - i$ largest jobs in $B_a$. Note that in keeping with the discussion following Lemma 3.2 we have assigned the largest job in $B_a$ (the job which MA would select) the highest numbered node and the smallest early set: $E_{a+b} = E_a$.
6. Return to step 2.
7. We have reached the first terminal node. $L_a$ is a set of tardy jobs of minimum cardinality: the set that MA produces. Note $|L_a|$, and record it as $|L|$. Compute the minimal $F|L'_a$ using Theorem 2.1. This is the first feasible solution. It is usually a good solution, often optimal. If this is sufficient, the algorithm may terminate here. Otherwise, we use the solution as the current upper bound, UB, for the mean flow time, and continue.
8. For all nodes $a$ on the node list, add to $E_a$ all jobs $i$ such that $d_i > d_j$, where $j$ was found to be tardy in step 3, on the last pass through.
9. Choose the next node, $a$, for expansion as in step 2. If no nodes remain, we are done: the current UB is the optimal solution.
10. If $|L_a| = |L|$, check whether a tardy job, $j$, can be found as in step 3. If so, reject this node, and return to step 9. If no job is tardy, compute min $F|L'_a$ using Theorem 2.1. If less than UB, this solution becomes the new UB.

11. If $|L_a| < |L|$, compute lower bound (LB) using $E_a$ in Theorem 2.1.
    (a) If LB $\geq$ UB, reject this node, and eliminate from the node list any other remaining nodes emanating from the same node, $\alpha$, that $a$ comes from; that is, eliminate any $a' \epsilon B_a$, where $a \epsilon B_a$. Return to step 9.
    (b) If LB $<$ UB, expand node $a$ as in steps 3, 4, and 5. Return to step 9.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $p_i$ | 3 | 8 | 7 | 9 | 4 | 6 | 13 | 10 |
| $d_i$ | 11 | 12 | 16 | 20 | 25 | 28 | 32 | 39 |

1. $E_1 = \{1, 5, 6\}$.
2. $a = 1$.
3. $j = 3$; $J_a = \{1, 2, 3\}$.
4. 2 $D$ 1 and 3, so $B_1 = \{2\}$.
5. $L_2 = \{2\}$, $E_2 = \{1, 5, 6\}$.
2. $a = 2$.
3. $j = 6$; $J_a = \{1, 3, 4, 5, 6\}$.
4. $B_2 = \{3, 4\}$
   etc.

As an example, consider:

In Figure 1, the resulting branch-bound network is given. Each nonterminal node, $a$, is labeled at the right with its early set $E_a$. The arrows are labeled with the jobs assigned to the late set, so that the labels on the arrows leading to $a$ make up $L_a$. Terminal nodes have associated min $F$ entered below; lower bounds on $F$ are at left of other nodes.

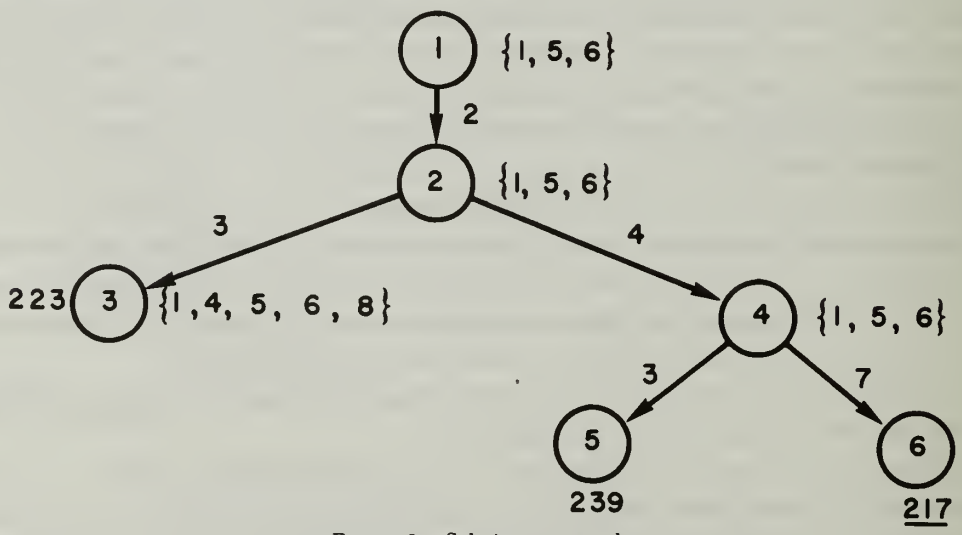


FIGURE 1. Solution to example.

The optimal late set is the one MA gives: $L = \{2, 4, 7\}$, and the optimal schedule using Theorem 2.1 is (1, 5, 3, 6, 2, 8, 4, 7) with $F = 217$.

## 5. COMPUTATIONAL RESULTS

The algorithm has been programmed in Fortran V on the Univac 1108. Twenty-five different problems have been run, five each for $n=10$, 50, 75, 100, and 150 jobs. Processing times were uniformly distributed between 1 and 10, and the due date for each job was set equal to its processing time plus a number uniform in $(1, 2n)$. This gives an *average tardiness factor* [7]

$$t=1-\mu_d/n\mu_p$$

in the vicinity of 0.7 or 0.8 for all problems, where $\mu_d$ and $\mu_p$ are the mean due date and processing time, respectively. This factor, which roughly indicates the average fraction of jobs that will be late, has been found to have a major influence on the performance of similar algorithms (see [1], [7]), with $t=0.6$ to 0.8 found to give the more difficult problems.

For each problem, the first feasible solution or "Moore's Solution," obtained by using MA in conjunction with Theorem 2.1, was computed along with the CPU time required to get it. The optimal solution was then found, and its time noted. The results are summarized in Table 1. Besides the average CPU times, the number of times that Moore's solution turned out to be optimal is given, as well as the percent error, both average and worst case, that would be occasioned by using Moore's solution, computed as

$$100 \cdot [(\text{Moore's Solution}) - (\text{Optimal Solution})]/(\text{Optimal Solution}).$$

Several points about the computational results are noteworthy. First, the variance in computer times to solve problems of a given size (not given in Table 1) is remarkably small. For the 10-job problems, the CPU times to find the optimal solution ranged from 2.52 to 2.68 seconds, while for the 150-job problems the range was 30.63 to 34.30 seconds. All the variances were similarly small, the largest coefficient of variation (standard deviation over mean) being 0.09 for the 75-job problems. For this reason, the small number of runs in each category was deemed sufficient.

TABLE 1. *Computational Results*

| Number of jobs per problem | 10 | 50 | 75 | 100 | 150 |
|---|---|---|---|---|---|
| Number of problems run | 5 | 5 | 5 | 5 | 5 |
| Mean CPU time (secs.) to Moore's Solution | 2.5 | 3.1 | 3.9 | 5.2 | 9.9 |
| Mean CPU time (secs.) to optimal Solution | 2.6 | 4.0 | 6.8 | 12.0 | 32.6 |
| Number of Moore's Solutions which were optimal | 4 | 1 | 2 | 2 | 1 |
| Mean error using Moore's Solution (%) | 1.8 | 1.0 | 0.6 | 0.3 | 1.1 |
| Maximum error using Moore's Solution (%) | 9.1 | 2.2 | 1.7 | 1.0 | 2.1 |

We see that the Moore's Solution is "usually" optimal for "small" problems, "sometimes" optimal for "large" problems, and in any case gives solutions within 1 or 2 percent of the optimal. On the

other hand, the additional computational effort to continue to optimality is remarkably little, even for the largest problems solved. Larger problems were not investigated because core storage capacity would have been exceeded, as the program was written.

## 6. ACKNOWLEDGEMENT

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# OPTIMAL PRODUCTION GROWTH FOR THE MACHINE LOADING PROBLEM

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**ABSTRACT**

This paper investigates a production growth logistics system for the machine loading problem (generalized transportation model), with a linear cost structure and minimum levels on total machine hours (resources) and product types (demands). An algorithm is provided for tracing the production growth path of this system, viz. in determining the optimal machine loading schedule of machines for product types, when the volumes of (i) total machine hours, and (ii) the total amount of product types are increased either individually for each total or simultaneously for both. Extensions of this methodology, when (i) the costs of production are convex and piecewise linear, and (ii) when the costs are nonconvex due to quantity discounts, and (iii) when there are upper bounds for productions are also discussed. Finally, a "goal-programming" production growth model where the specified demands are treated as just goals and not as absolute quantities to be satisfied is also considered.

## 1. INTRODUCTION

Growth path logistics system and the prespecified market growth rate in the context of a "transportation problem" was analyzed by Srinivasan and Thompson [14] and by Fong and Rao [11]. These are general extensions and characterizations of earlier work on such logistics systems modelled as a "transportation problem" [8, 9]. However, nothing has been reported into the literature on such logistics systems for the "machine loading problem" [1, 7, 10] characterized as a generalized transportation problem or the analogous "networks with gains" [13]. This paper is expected to fill this gap in considering a logistics system modelled as a "generalized transportation problem" for the machine loading problem where one minimizes the total production costs of a homogeneous good given by different product types from a set of machines. Given a planning horizon, we investigate how the optimal total cost, the machine hours available (treated as sources), and product types required (considered as demands), vary continuously as a path, when the total volume in the system, both with respect to total machine hours available and total amounts required of each product type and their totals increase. Such a 'growth path' analysis is essential for optimal production planning for both short and long term projections of the entire system relative to facilities, financial commitments, growth of the firm, manpower needs, etc.

Such an investigation is needed in the context of production-marketing and logistics system. For example, consider a firm that plans to expand so that it has to ascertain where capacities are to be increased and whose demands should be sought more. It is reasonable to assume that the firm wishes to maintain at least its current levels of production and demands. The cost of production of a product type from a specified machine involves also manpower, transportation, promotion and other related costs and the revenue from selling the same. It is assumed that the net costs (or revenues) are

*proportional* to the amounts concerned, to enable us to obtain a net per unit cost (or negative per unit profit) for each machine type-product type combination. For such a case, the growth path will provide the effect on the optimal solution as the total volume of each type mentioned is increased. Further, job shops that are committed, comprising the machine types involved and the product types produced by these machines with the associated personnel, would benefit by deciding on the additional workforce required in overtime hiring or redistribution of work schedule decisions when lay-off is contemplated.

## 2. THE MACHINE LOADING PROBLEM AND AN OPERATOR THEORY FOR OPTIMAL PRODUCTION PATH DECISIONS

The following provides the mathematical formulation of the "lower bounded and constrained total volume machine loading problem."

(1) Let $\mathscr{I}=\{1, 2, \ldots, i, \ldots, m\}$ the set of machine types (services), and

(2) $\mathscr{J}=\{1, 2, \ldots, j, \ldots, n\}$ the set of product types (demands).

For $i\epsilon\mathscr{I}$ and $j\epsilon\mathscr{J}$, the following are defined

$x_{ij}$ = amount of product type $j$ to be produced utilizing machine type $i$ $(\geq 0)$.

$c_{ij}$ = per unit cost of production of product type $j$ using unit time of machine type $i$ $(\geq 0)$.

$b_j$ = minimum amount of product $j$ required $(\geq 0)$.

$a_i$ = current availability of machine time units of machine type $i$ $(\geq 0)$.

$e_{ij}$ = machine utilization time of machine type $i$ per unit of product $j$ $(>0)$.

$S_i(T_j)$ = surplus amount of machine hours available of the type $i$ (demanded of product type $j$) $(\geq 0)$.

$g_i(h_j)$ = unit cost of production using one unit time of machine type $i$ (or one unit of product type $j$) $(\geq 0)$.

With these known values, the following problem (P) is defined.

(3) $$\text{Minimize } Z=\sum_{i\epsilon\mathscr{I}}\sum_{j\epsilon\mathscr{J}} c_{ij}x_{ij}+\sum_{i\epsilon\mathscr{I}} g_i(a_i+S_i)+\sum_{j\epsilon\mathscr{J}} h_j(b_j+T_j),$$

subject to the following constraints:

(4) $$\sum_{j\epsilon\mathscr{J}} e_{ij}x_{ij}\leq a_i+S_i \quad \text{for } i\epsilon\mathscr{I},$$

(5) $$\sum_{i\epsilon\mathscr{I}} x_{ij}=b_j+T_j \quad \text{for } j\epsilon\mathscr{J},$$

(6) $$x_{ij}\geq 0,\ S_i\geq 0,\ T_j\geq 0 \quad \text{for } i\epsilon\mathscr{I} \text{ and } j\epsilon\mathscr{J},$$

where $x_{ij}$, $S_i$ and $T_j$ are the decision variables.

REMARK 1: We can generalize this problem in the framework of netowrk with gains [13] which can also accommodate a coefficient $f_{ij}$ in constraint set (5) for each $x_{ij}$. For such a case one could define $x'_{ij}=f_{ij}x_{ij}$ so that new constants $c'_{ij}=c_{ij}/f_{ij}$ and $e'_{ij}=e_{ij}/f_{ij}$ change this problem to have the same

structure as in (3)–(6). Thus without any loss of generality we consider problem $P$ where one always ensures unity as coefficients of $x_{ij}$ in constraint set (5). Further, if we have upperbounds for $x_{ij}$, then the problem can be handled by the method provided by Balachandran and Thompson [1].

The problem $P$ with $S_i=0$ and $T_j=0$ may not have a feasible solution. Thus we will introduce a slack column $n+1$, with $e_{i,n+1}=1$, $c_{i,n+1}=0$ for $\forall\, i\in\mathscr{I}$ and a fictitious machine type with abundant availability $a_{m+1}$ and a prohibitive per unit cost of production $c_{m+1,j}=M$ (a high cost) with $e_{m+1,j}=1$ for $\forall\in\mathscr{J}$. Further let $c_{m+1,n+1}=0$; $e_{m+1,n+1}=1$ and *no constraint exists for the newly created column* $n+1$. Let us now define the index sets $I=\mathscr{I}\cup\{m+1\}$, $J=\mathscr{J}\cup\{n+1\}$. (One may put $g_{m+1}$ to be $M$ and $h_{n+1}=b_{n+1}=S_{m+1}=T_{n+1}=0$, but since that adds only a constant to the objective function (3) these, are not specified here.)

Thus the resulting problem will be

$$\text{Min } Z=\sum_{i\in I}\sum_{j\in J}c_{ij}x_{ij}+\sum_{i\in I}g_i(a_i+S_i)+\sum_{j\in J}h_j(b_j+T_j), \tag{7}$$

subject to

$$\sum_{j\in J}e_{ij}x_{ij}=a_i+S_i \qquad \text{for } i\in I, \tag{8}$$

$$\sum_{i\in I}x_{ij}=b_j+T_j \qquad \text{for } j\in\mathscr{J}, \tag{9}$$

$$x_{ij}\geq 0; \quad \text{for } i\in I, j\in J; \quad S_i\geq 0 \quad \text{for } i\in\mathscr{I}; \quad T_j\geq 0 \quad \text{for } j\in\mathscr{J}, \quad \text{and } S_{m+1}=T_{n+1}=b_{n+1}=h_{n+1}=0. \tag{10}$$

$$\text{Let } K_M=\sum_{i\in I}\left(\sum_{j\in J}e_{ij}x_{ij}\right) \text{ and} \tag{11}$$

$$K_P=\sum_{j\in\mathscr{J}}\left(\sum_{i\in I}x_{ij}\right). \tag{12}$$

Here in (7)–(11), $x_{ij}$, $S_i$ and $T_j$ are the decision variables. Constraint (8) for every $i\in I$ states that the total time provided by machines of type $i$ for production of all product types, should be equal to the total available time units $a_i+S_i$. Since $S_i\geq 0$ from (10), the total available time is at least equal to the current level of operation $a_i$ (the lower bound at $i$th row). This type of statement holds for columns $j\in\mathscr{J}$. (Note we don't have a constraint for column $(n+1)$). The production system given here has two different total volumes, one $K_M$ the *total machine hours* available and $K_P$ the *total amount of all products* required in the system. The objective function minimizes the total cost in the system (production costs, securing additional time and demands). Though the objective function (7) can be rewritten from Equations (8) and (9) as

$$\text{Min } Z=\sum_{i\in I}\sum_{j\in J}c'_{ij}x_{ij} \qquad \text{where} \qquad c'_{ij}=c_{ij}+g_ie_{ij}+h_j, \tag{13}$$

the form given in (7) is to be preferred since it is more suitable when the costs of producing from the

machines and meeting more demands are not necessarily proportional. In such a situation we represent $g_i(\cdot)$ and $h_j(\cdot)$ as functions of $(a_i+S_i)$ and $(b_j+T_j)$, respectively.

Since in $(m+1)$th row, $c_{m+1,j}=M$ a high penalty cost and $a_{m+1}$ is a very large availability, one is always assured of a feasible solution to problems (7)–(10) [10]. However if $x_{m+1,j}>0$, this shows there is no feasible solution to the original problem. The condition $x_{m+1,j}=0$ for $j\epsilon\mathcal{J}$ is necessary and sufficient for the existence of a feasible solution to the problem (3)–(6). Further $x_{m+1,n+1}$ is always $>0$ (see [10]).

By summing Equations (8) for $i\epsilon I$ and subtracting from (11), we can rewrite equation (11) by

$$K_M-\sum_{i\epsilon I} a_i=\sum_{i\epsilon I} S_i. \tag{14}$$

From (10), where $S_i\geq 0$, it follows that

$$K_M\geq \sum_{i\epsilon I} a_i=K_a \qquad \text{(say)}. \tag{15}$$

Similarly for the columns, from (9), (10), and (12) we have

$$K_P-\sum_{j\epsilon J} b_j=\sum_{j\epsilon J} T_j, \text{ so that} \tag{16}$$

$$K_P\geq \sum_{j\epsilon J} b_j=K_b \qquad \text{(say)}; \qquad (\text{note } b_{n+1}=T_{n+1}=0). \tag{17}$$

Again feasibility is always assured, due to $(m+1)$th row with large $a_{m+1}$; however, the presence of $x_{m+1,j}$ for any $j\epsilon\mathcal{J}$ is sufficient to show that the *original problem* has no feasible solution. Such positive $x_{m+1,j}$ provide the fact about those demands $j$, which are not satisfied, since they use up the machine hours from a fictitious machine.

We will now transform the problem given by (7)–(12) to a Standard Generalized capacitated transportation problem [1], with $(m+2)$ rows (machine types) and $(n+2)$ columns (product types). Let us add a large positive constant $N$ to both sides of (8) and (9) and rearrange, to obtain the following:

$$\sum_{j\epsilon J} e_{ij}x_{ij}+(N-S_i)=a_i+N \qquad \text{for } i\epsilon I \text{ and} \tag{18}$$

$$\sum_{i\epsilon I} x_{ij}+(N-T_j)=b_j+N \qquad \text{for } j\epsilon\mathcal{J}. \tag{19}$$

Similarly (7), (14), and (16) can be arranged, respectively, as

$$Z=\text{Min}\sum_{i\epsilon I}\sum_{j\epsilon J} c_{ij}x_{ij}-\sum_{i\epsilon I} g_i(N-S_i)-\sum_{j\epsilon J} h_j(N-T_j)+\sum_{i\epsilon I} g_i(N+a_i)+\sum_{j\epsilon J} h_j(N+b_j), \tag{20}$$

$$\sum_{i\epsilon I}(N-S_i)=(m+1)N+\sum_{i\epsilon I} a_i-K_M, \text{ and} \tag{21}$$

$$\sum_{j\in J}(N-T_j)=(n+1)N+\sum_{j\in J}b_j-K_P. \tag{22}$$

Let the following definitions be made:

$$x_{i,n+2}=N-S_i;\ e_{i,n+2}=1;a_i'=a_i+N \text{ and } c_{i,n+2}=-g_i \qquad \text{for } i\in I; \tag{23}$$

$$x_{m+2,j}=N-T_j;\ e_{m+2,j}=1b_j'=b_j+N \text{ and } c_{m+2,j}=-h_j \qquad \text{for } j\in J; \tag{24}$$

$$a_{m+2}=(n+1)N+\sum_{j\in J}b_j-K_P; \qquad b_{n+2}=(m+1)N+\sum_{u\in I}a_i-K_M; \tag{25}$$

$$c_{m+2,n+2}=M \qquad \text{(a large positive number);} \tag{26}$$

$$Z_0=\sum_{i\in I}g_i(N+a_i)+\sum_{j\in J}h_j(N+b_j)=a \text{ constant;} \tag{27}$$

$$I'=I\cup\{m+2\},\ J'\cup\{n+2\}, \qquad \text{and } \mathscr{J}'=\mathscr{J}\cup\{n+2\}. \tag{28}$$

With these definitions given by (23)–(28), (20)–(22), (18) and (19), the problem (7)–(10) can be seen to be equivalent to the following:

$$\text{Minimize } Z=\sum_{i\in I'}\sum_{j\in J'}c_{ij}x_{ij}+Z_0, \tag{29}$$

Subject to the following constraints:

$$\sum_{j\in J'}e_{ij}x_{ij}=a_i' \qquad \text{for } i\in I', \tag{30}$$

$$\sum_{i\in I'}x_{ij}=b_j' \qquad \text{for } j\in\mathscr{J}', \tag{31}$$

$$x_{ij}\geq 0, \qquad \text{for } i\in I', \qquad \text{and } j\in J'; \tag{32}$$

$$x_{i,\,n+2}\leq N, \qquad \text{for } i\in I, \tag{33}$$

$$x_{m+2,j}\leq N, \qquad \text{for } j\in J. \tag{34}$$

REMARK 2: Since $M$ is assumed to be large positive number, $c_{m+2,\,n+2}=M\Rightarrow x_{m+2,\,n+2}=0$ in any optimal solution. Further the constraints $x_{m+2,\,j}\geq 0$ and $x_{i,\,n+2}\geq 0\Rightarrow S_1\leq N$ and $T_j\leq N$, respectively which are always true by our choice of $N$.

Now we can think of two possible growths in the system, viz. when $K_P$ the total volume of products increase from $K_P$ to $K_P+\delta$. From (23)–(27), this is equivalent to studying the effects of changing

$b_{n+2}$ to $b_{n+2}-\delta$. (Similarly for machine growth.) Since we know the current production schedule has solved the problem where all $S_i=0$ and $T_j=0$, we have the optimal solution $x_{ij}$ and $Z$ with $K_P=K_b$ and $K_M=K_a$. Then the optimal solution for any other $K_M$ or $K_P$ can be obtained via parametric programming specially developed as "Rim Operators" by Balachandran and Thompson [2, 4] which takes advantage of the structure of the Generalized Transportation Problem. Specifically changes in $K_M$, starting with $K_a$, can be completely analyzed (since $K_M=K_a+\delta$) by utilizing the cell rim operator $\delta$ $R_{\bar{m}+2,\,n+1}$, where $\delta=K_M-K_a$. This cell rim operator provides the optimum solution of a problem where the entire data of $P$ are used except $a_{\bar{m}+2}=a_{m+2}-\delta$, from the known optimal solution of $P$. (Essentially a parametric analysis when just the $(m+2)$th row value is changed from $a_{m+2}$ to $a_{m+2}-\delta$.) Similarly for changes in $K_P$ starting with $K_b$ can be completely analyzed with the known optimal solution of $P$ by utilizing the cell rim operator $\delta$ $R_{\bar{m}+1,\,n+2}$ where, now, $\delta=K_P-K_b$. (The $n+2$th column total, changes from $b_{n+2}$ to $b_{n+2}-\delta$.) Constructive algorithms and solution procedures for all rim operators for *all* values of $0\leqslant\delta\leqslant\infty$ are given in [2, 4]. Further the simultaneous changes of both $K_M$ and $K_P$ also can be analyzed by the cell operator $\delta$ $R_{\bar{m}+2,\,n+2}$.

To start with, we have an optimal solution to the "original" problem where $\sum_{i\in I} S_i=0$ and $\sum_{j\in J} T_j=0$ (in other words, for $K_M=K_a$ and $K_P=K_b$). Let us call this as the "original optimum solution" to problem (29)–(34). From (23) and (24) we know that $x_{i,\,n+2}=N$ for $i\in I$ and $x_{m+2,\,j}=N$ for $j\in J$ and $x_{m+2,\,n+2}=0$ from Remark 2. Substituting these values in (29)–(34) where $K_M=K_a$ and $K_P=K_b$, we have

$$\text{Min } Z=\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}+\Bigg[\underbrace{Z_0-N\Big(\sum_{i\in I} g_i+\sum_{j\in J} h_j\Big)}_{\text{(constant)}}\Bigg], \tag{35}$$

subject to

$$\sum_{j\in J} e_{ij}x_{ij}=a_i \qquad \text{for } i\in I, \tag{36}$$

$$\sum_{i\in I} x_{ij}=b_j \qquad \text{for } j\in \mathscr{J}, \text{ and} \tag{37}$$

$$x_{ij}\geqslant 0 \qquad \text{for } i\in I \text{ and } j\in J. \tag{38}$$

The problem (35)–(38) is a "generalized transportation problem" and can be solved by four index algorithm [5] developed by Balachandran and Thompson. Other procedures are also available, e.g., see Balas [7], Glover and Klingman [12].) This "original optimum solution" for the problem (35)–(38) which is uncapacitated, together with the new values for $x_{i,\,n+2}$ and $x_{m+2,\,j}$ provide an optimum solution to (29)–(34). The above set of solutions for $x_{ij}$, $i\in I'$ and $j\in J'$ will be called hereafter as a "basic optimum solution." The following theorem provides a constructive procedure for generating such a "basic optimum solution," from the "original optimum solution" for problem (29)–(34).

THEOREM 1: Let $B$ denote the optimum basis for (35)–(38) with $u_i$ and $v_j$ as the dual variables (see for duals [1, 2]). Let the indices $r$, $s$ be defined such that

$$v_{n+2}=-(u_r+g_r)=\max_{i\in I}\,[-(u_i+g_i)] \text{ and} \tag{39}$$

(40) $$u_{m+2} = -(v_s + h_s) = \max_{j \in J} [-(v_j + h_j)].$$

Then $B' = B + \{(r, n+2)\} + \{(m+2, s)\}$ is an optimal basis to the problem (29)–(34) with the dual solutions $u_i$ for $i \in I'$ and $v_j$ for $j \in \mathcal{J}'$.

PROOF: First, let us prove that $B'$ as given above is a basis. Since $B$ is a basis for the problem (35)–(38) with $(m+1)$ rows and $(n+1)$ columns, this basis graph will be a "one-forest" consisting of $(m+n+1)$ cells and mutually disconnected "one-trees" [1, 7]. (A tree with an extra edge or a tree with the node for $(n+1)$th column). Each row and column contains at least one cell. Since $(r, n+2)$ and $(m+2, s)$ are cells in $B'$, they provide one cell in each $(m+2)$th row and $(n+2)$th column. Since cell $(r, n+2)$ is unique in column $(n+2)$ and cell $(m+2, s)$ is unique in row $(m+2)$, these cells, when they are joined as arcs in the one trees where $r$th row is present and $s$th column is present, respectively, will still preserve the one tree property. Further the mutual disconnectedness existed in the "one-forest" of the original $B$ is also preserved. Thus $B'$ also possess the one-forest property. Hence $B'$ is a basis. If $(i, j) \notin B'$, then $x_{ij} = 0$ or take its upper bound. Let $LB'$ and $UB'$ be the set of cells where $x_{ij} = 0$ and those cells that take the upper bounds, respectively. Let the basis structure be defined as $(B', LB', UB')$ for the problem.

Now to prove that $B'$ is optimal, we should show that [1, 7, 10]

(41) $$(i, j) \in B' \Rightarrow c_{ij} = e_{ij} u_i + v_j,$$

(42) $$(i, j) \in LB' \Rightarrow c_{ij} \geq e_{ij} u_i + v_j, \text{ and}$$

(43) $$(i, j) \in UB' \Rightarrow c_{ij} \leq e_{ij} u_j + v_j.$$

Since $B$ is optimal to (35)–(38), (41) holds for the cells in $B$. For the cell $(r, n+2)$, $c_{r, n+2} = -g_r$ from (23). Further from (39) $-g_r = u_r + v_{n+2}$ so that $c_{r, n+2} = u_r + v_{n+2}$. Since $e_{r, n+2} \equiv 1$ from (23),

$$c_{r, n+2} = e_{r, n+2} u_r + v_{n+2}$$

thus satisfying (41). A similar argument holds for the cell $(m+2, s)$. For the cells $(i, j) \in LB$ (42) holds since $B$ is optimal. We constructed the solution to (29)–(34) so that $x_{i, n+2} = N$ for $i \in I$ and $x_{m+2, j} = N$ for $j \in J$. Hence these cells are at their upper bounds, or $\in UB'$. For $(i, n+2) \in UB'$, with $i \in I$, from (39) we have

$$-(u_i + g_i) \leq v_{n+2} \qquad \text{so that } c_{i, n+2} = -g_i \leq u_i + v_{n+2} \qquad (\text{since } e_{i, n+2} = 1)$$

thus satisfying (43) for such $(i, j) \in UB'$. Similar statement is true for cells $(m+2, j)$ for $j \in J$. Since $c_{m+2, n+2} = M$ from (26), (42) trivially holds. Further $(i, j) \in LB$ satisfies (42). These cells together with cell $(m+2, n+2)$ make $LB'$ and thus (42) holds for all $(i, j) \in LB'$. Q.E.D.

These results are summarized as an algorithm given below. (Algorithm A1.) First we find an "original optimal solution" using our code [5] where $K_M = K_a$ and $K_P = K_b$. Later the following four options are investigated.

(i) $K_a \leq K_M \leq K_a + N$; (ii) $K_b \leq K_P \leq K_b + N$; (iii) Both variations in $K_M$ and $K_P$ simultaneously where if $K_M = K_a + \delta$; $K_P = K_b - \delta$ and if $K_P = K_b + \delta$; $K_M = K_a - \delta$ and (iv) $K_M$ and $K_P$ both increase at the rate of $\delta\, \alpha_M$ and $\delta\, \alpha_P$.

ALGORITHM A1: To find the production growth by solving for optimal solution of (29)–(34) for all options (i) through (iv) given above.

STEP 1: Find the "original optimal solution".

Solve the problem given by (35)–(38) by using the "generalized stepping stone method" [1, 9] (see Algorithm A4 of [1]). Let $Z$ denote the optimum value with basis as $B$. Those which are nonbasic, i.e., $x_{ij} = 0$, constitute the set of lower bounds called $LB$. Utilizing this optimal solution to (29)–(34) as the *same* for all $i \epsilon I$ and $j \epsilon J$, define $x_{i,n+2} = N$ for $i \epsilon I$; $x_{m+2,j} = N$ for $j \epsilon J$ and $x_{m+2,n+2} = 0$. Evaluate the duals $v_{n+2}$ and $u_{m+2}$ using (39) and (40) after identifying the indices $r$ and $s$. Define the new basis as $B' = B + \{r, n+2)\} + \{m+2, s)\}$ as the basis for (29)–(34).

OPTION 1: Production growth when extra machine hour capacities are available.

Application of Rim Operator $\delta\, R^{-}_{m+2,\,n+1}$ for any $0 \leq \delta \leq N$.

STEP 2: Let the iteration count $K = 1$; $X_1 = X$, $B_1 = B'$; $Z_1 = Z$, $u_{i1} = u_i$ for $i \epsilon I'$, $v_{j,1} = v_j$ for $j \epsilon J'$.

STEP 3: With the solution of Step 2, apply the cell rim operator $\delta\, R^{-}_{m+2,\,n+1}$ as given by Algorithms A8, A15, of the rim operator [2, 4] and identify the discrete set of "corner points", $0 = \mu_0, \mu_1, \mu_2, \ldots, \mu_i, \ldots, \mu_t$ such that

(i) for every interval

$$\sum_{i=0}^{j-1} \mu_i < \delta \leq \sum_{i=0}^{j} \mu_i$$

the basis $B_j$ doesn't change, for $j = 0$ to $t$ and

(ii) the last interval contains $N$, such that

$$\sum_{i=0}^{t-1} \mu_i < N \leq \sum_{i=0}^{t} \mu_i,$$

so that the optimal solution $Z_j$ and the corresponding basis $B_j$ giving $X = \{x_{ij}\}$ are completely determined for $0 \leq \delta \leq N$. This is equivalent to analyzing the production growth $K_M$ when $\delta$ moves from $K_a$ to $K_a + N$. The exact calculations procedures are given in Algorithms A8, A15 of [2, 4]. (A detailed algorithm for the transportation type problem is given in [14].)

OPTION 2: Production Growth when extra demands for products are increasing from the current level $K_b$ to $K_b + N$.

Application of cell Rim Operator $\delta\, R^{-}_{m+1,\,n+2}$ for any $0 \leq \delta \leq N$. Steps (2) and (3) are similar to those of option (1) except we apply cell rim operator $\delta\, R^{-}_{m+1,\,n+2}$ and get a new set of "corner points" $\mu_i$. This growth path yields the study of $K_P$ when $\delta$ moves from $K_b$ to $K_b + N$.

OPTION 3: Simultaneous variation in $K_M$ and $K_P$ such that (i) when $K_M = K_a + \delta$, $K_P = K_P - \delta$ or (ii) when $K_P = K_P + \delta$, $K_M = K_a - \delta$.

Application of cell rim operator $\delta\, R\bar{\mu}_{m+2,\,n+2}$ for any $0 \leq \delta \leq N$, $-$ or $+$ as (i) or (ii) is required.

Again Steps (2) and (3) are similar to those of option (i) except we now apply cell rim operator $\delta\, R^{\mp}_{m+2,\,n+2}$ and obtain a new set of "corner points" $\mu_i$ and their associated basis $B_i'$, $Z$ value and solu-

ion $X=\{x_{ij}\}$ depending upon whether we want to analyze (i) or (ii), respectively, so that (i) when $K_a \leq K_M \leq K_a+N$, then $K_b \geq K_P \geq K_b-N$ or (ii) when $K_b \leq K_P \leq K_b+N$, then $K_a \geq K_M \geq K_a-N$.

OPTION 4: Simultaneous *increase* in $K_M$ and $K_P$ in the directions (steps) of $\alpha_M$ and $\alpha_P$ such that $K_M=K_a+\alpha_M\delta$ and $K_P=K_b+\alpha_P\delta$.

Here we *increase* both $K_a$ and $K_P$ so that there is growth on both volumes in the system (viz. in machine capacity as well as required demands). This is achieved by applying area rim operator $\delta R^A$ such that the $(m+2)$th row total is *decreased* by $\delta\alpha_M$ and $(n+2)$th column total is *decreased* by $\delta\alpha_P$. This can be achieved by using Algorithms A8, A15 of [2, 4]. Once again Steps (2) and (3) are the same except for this area rim operator application, and one can get the "corner points" $\mu_i$'s such that the basis $B_t$, associated $Z$ value and solution $X=\{x_{ij}\}$ can be obtained systematically.

REMARK 3: It is to be noted that each cell rim operator is independent (see [2]), so that for Option 4, one could apply Option 1 and to the resultant problem can apply Option 2, or vice versa, so that results for simultaneous growths can be obtained. This avoids area rim operator application. Thus a rectangular array of "corner points" can be obtained. For instance the first interval $\mu_0 < \delta \leq \mu_1$ is obtained utilizing $\delta\ R^-_{m+2,\,n+1}$ cell rim operator giving the new basis, $Z$ value and solution X. To this resultant problem one can apply Option 2 in its entirety so that $K_b \leq K_P \leq K_P+N$ is investigated for $K_a \leq K_M \leq K_a+\mu_1$. Then we go to the next interval for $\delta\ R^-_{m+2,\,n+1}$ such that analysis for $K_a+\mu_1 \leq K_M \leq K_a+\mu_1+\mu_2$ is obtained and the entire Option (2) is once again obtained and so on until we cover the entire range of $K_a \leq K_M \leq K_a+N$.

REMARK 4: For any $K_M$ (or $K_P$) such that $K_a \leq K_M \leq K_a+N$ we know the value of $\delta=K_M-K_a$ (or $K_P-K_b$) and an index $t$, such that

$$\sum_{i=0}^{t-1} \mu_i < \delta \leq \sum_{i=0}^{t} \mu_i \text{ and} \tag{44}$$

$$\lambda=\delta-\sum_{i=0}^{t-1} \mu_i. \tag{45}$$

The rim operator Algorithm A8 and A15 [2, 4] provides a procedure of updating $x_{ij}$ for the cells in the two-tree, such that

$$x_{ij}\ (\text{new}) = x_{ij,\,t}+m_{ij,\,t}\lambda \tag{46}$$

where $x_{ij,\,t}$ are the solution $X=\{x_{ij}\}$ at the $t$th interval, $m_{ij,\,t}$ the multipliers obtained from Algorithm A8, A15 [2, 4] which are nonzero of the cells in the two-tree and $\lambda$ defined in (45). The associated total cost $Z$ will be

$$Z\ (\text{new}) = Z_t-\lambda\ (u_{m+2,\,t}+v_{n+1,\,t}), \tag{47}$$

where $u_{i,\,t}$ and $v_{j,\,t}$ are the duals obtained for $i$th row, and $j$th column for the $t$th interval. (Note that $e_{m+2,\,j} \equiv 1$ for $(m+2)$th row and $v_{n+1,\,t} \equiv 0$ for all $t$ since there is no constraint in $(n+1)$th column.)

The recurrence relation for each interval $k$ will be

$$Z_{k+1}=Z_k-\mu_k(u_{m+2,k}+v_{n+1,k}) \qquad \text{(note, } v_{n+1,k}\equiv 0), \tag{48}$$

$$x_{ij,k+1}=x_{ij,k}+m_{ij,k}\mu_k, \text{ and} \tag{49}$$

$$u_{i,k+1}=u_{i,k}+m_{i,k}\Delta \qquad v_{j,k+1}=v_{j,k}+n_{j,k}\Delta, \tag{50}$$

where $\Delta$ is found in Algorithm A15, [4] and $m_i$, $n_j$ are the multipliers for the duals (see Algorithm A9 of [2]).

Thus the Equations (48)–(50) provide an updating recurrence relation for $(k+1)$th interval where $k$th interval values for $Z$, $X$, and $u$, $v$ are known. We know the first interval value, i.e., when $k=0$.

All these discussions hold for every option.

From (47) and (48) it follows that the cost $Z$ for any $0\leq\delta\leq N$ is given by

$$Z=Z_1-\sum_{k=1}^{t-1}[\mu_k(u_{m+2,k}+v_{n+1,k})]-\lambda(u_{m+2,t}+v_{n+1,t}), \tag{51}$$

where $t$ and $\lambda$ are given by (44) and (45). Thus the rate at which the cost $Z$, increases when $\delta$ is increased is given by $-u_{m+2,k}-v_{n+1,k}$. This marginal cost is thus the negative sum of the duals for $(m+2)$th row and $(n+1)$th column. We also note that $u_{m+2,k}\leq 0$ and $v_{n+1,k}\equiv 0$ for the generalized transportation problem (see [1]). Thus $-u_{m+2,k}\geq 0$ and so $u$ is a *nondecreasing* function of $k$ for the operator $\delta\ R^{-}_{m+2,n+1}$ so that the marginal cost is nondecreasing in $\delta$ and thus also in $K_M=K_a+\delta$.

Now, in the case of Option 2 we go to the operator $\delta\ R^{-}_{m+1,n+2}$ and Equation (47)–(50) could be modified by putting $m+1$ for $m+2$th row and $n+2$ for $n+1$th column. Now since the "absorbing cell" $(m+1, n+1)$ is always on the basis [1], $u_{m+1}\equiv 0$ and all $v_{j,t}\geq 0$. Thus, the marginal cost will be a *nonincreasing function* of $\delta$ since $-v_{n+2,k}\leq 0$ for the operator $\delta\ R^{-}_{m+1,n+2}$. Thus it is nonincreasing in $K_P$ since $K_P=K_b+\delta$.

Similar discussions can be given for Options (3) and (4) depending upon the corresponding rim operator's properties. These results lead to the following theorem.

THEOREM 2: The optimal values for the variables $x_{i,n+2}$ for $i\epsilon I$ and $x_{m+2,j}$ for $j\epsilon J$ as determined by Algorithm A1 are nondecreasing functions of $\delta=K_M-K_a$ for Option (1) and nondecreasing functions of $\delta=K_P-K_b$ for Option 2.

PROOF: It follows from (44) and (45) that the proof of this theorem will be complete, if we show that it holds for

$$\sum_{k=0}^{t'-1}\mu_k\leq\delta\leq\sum_{k=0}^{t'}\mu_k. \tag{52}$$

With $\lambda$ defined in (45) for any $t'$, $0\leq t'\leq t$, it is enough if we show that (52) holds for $0\leq\lambda\leq\mu_t$. It is known that $x_{ij,t+1}$ as defined in (49) is the same as that given by (46) for $\lambda=\mu_t$.

From (46), $x_{ij}$ increases for the cells such that $m_{ij,t}$ are positive. Of course we maintain always $x_{m+2,n+2}=0$ due to huge cost. Thus the proof will depend on showing that no cell $(i, n+2)$ for $i\epsilon I$ or

$(m+2, j)$ for $j \epsilon J$ has a positive $m_{ij}$. To prove this assume the contrary that $m_{r,n+2} > 0$ where $r \epsilon I$. Since the "two-tree" associated with the rim operator [2] has exactly two cells in column $(n+2)$, which are $(r, n+2)$ and $(m+2, n+2)$, these are the adjacent cells in the "two-tree" and since $m_{ij}$'s are alternately positive and negative in the adjacent cells in the two-tree, both these specified cells cannot have positive $m_{ij}$'s. Since cell $(m+2, n+2)$ has positive multipliers $m_{ij}$, we know $(r, n+2)$ should have a negative multiplier. A similar argument holds for $(m+2, s)$ where $s \epsilon \mathcal{J}$ so that the theorem holds. Similar arguments are also applicable for Option (2).

## 3. EXTENSIONS AND GENERALIZATIONS

### 3.1 Capacitated Growth Paths

In certain machine loading problems, certain cells $x_{ij}$ for $i \epsilon I$ and $j \epsilon J$ are constrained by certain maximum capacities (upper bounds) $U_{ij}$, so that $0 \leq x_{ij} \leq U_{ij}$. In these cases, since our original rim operator theory was developed for the upper bounded generalized transportation problem [1, 2, 4], Algorithm A1 and Theorems 1 and 2 will be still true, though the Algorithm A1 has to be slightly modified to take care of $U_{ij}$'s.

### 3.2 Convex Costs of Production

The theory discussed in Section (2) can be used when the costs of production $g(S_i)$ and $h(T_j)$ are general convex functions that can be approximated by a convex piecewise linear functions. Consider, for example, Option (1) with changes in row total $a_1$ only. If $a_1 = 100$ and per unit cost of adding $a_1$ are as follows:

$$\$5/\text{unit for } 0 \leq S_1 \leq 5; \qquad \$6/\text{unit for } 5 < S_1 \leq 10; \qquad \$7/\text{unit for } S_1 > 10.$$

Then we create 3 fictitious machine types for the first, say $t = 1, 2, 3$; with the same $c_{ij}$'s for all $j \epsilon J$, and $t = 1, 2, 3$, but with $c_{1,n+2} = -5$, $c_{2,n+2} = -6$, and $c_{3,n+3} = -7$. Let $S_{1,1}$, $S_{1,2}$, and $S_{1,3}$ denote the surplus machine hours of type 1 in the ranges of $0 \leq S_{11} \leq 5$, $5 < S_{12} \leq 10$, $10 < S_{13}$ so that $S_1 = S_{1,1} + S_{1,2} + S_{1,3}$. Thus we define $x_{1,n+2} = 5 - S_{1,1}$, $x_{2,n+2} = (10-5) - S_{1,2}$, and $x_{3,n+2} = N - S_{1,3}$ to get upper bounded constraints (viz. $x_{1,n+2} \leq 5$; $x_{2,n+2} \leq 5$; and $x_{3,n+2} \leq N$). Further, due to convexity, the optimum solution forces, for example, $S_{1,2} > 0$ (due to $x_{2,n+2} < 5$) if and only if $S_{11} = 5$ (i.e., $x_{1,n+2} = 0$). Similarly $S_{1,3} > 0$ (i.e., $x_{3,n+2} < N$) if and only if $S_{11} = 5$ and $S_{12} = 5$. (One can use cost operators [2] for such an analysis instead of increasing the size of the problem.)

Further in the "Transportation Type Problems With Quantity Discounts" Balachandran and Perry [6] have provided an algorithm for an objective function which is neither convex nor concave. Thus even if $g(S_i)$ and $h(T_j)$ are nonconvex (nonconcave) then the above procedure could be utilized for such a growth model for the quantity discounted problem.

Again, if the growth rates for availability of machines or the product's demands increase at prespecified rates, say $\alpha_i$ for $S_i$ and/or $\beta_j$ for $T_j$ the model can be reformulated in this general framework with new objective function and right-hand sides for $(m+2)$th row and/or $(n+2)$th column appropriately modified. (Details similar to this are given in [14].)

### 3.3 Goal Programming Growth

It was imperative in problem (3)–(6) that the minimum demands $b_j$ and availabilities $a_i$ are necessarily met. On the contrary, it could be treated just as "goals" or as "expected values." Thus one may allow a *deficit* or surplus in each row or column which are nonnegative. Let $S_i^+ (S_{\bar{i}})$ and $T_j^+ (T_{\bar{j}})$ represent the surplus (deficit) at row $i$ and column $j$, respectively, so that $S^+ (S^-)$ and $T^+ (T^-)$ be their respective totals. Then we could assign a per unit reward (penalty) for the surplus (deficit) yielding the following "goal" programming [8] problem.

$$\text{(53)} \quad \text{Min } Z = \sum_{i\epsilon I}\sum_{j\epsilon J} c_{ij}x_{ij} + \sum_{i\epsilon I} g_i(a_i + S_i^+ - S_{\bar{i}}) + \sum_{j\epsilon J} h_j(b_j + T_j^+ - T_{\bar{j}}) + \sum_{i\epsilon I} (s_{\bar{i}}S_{\bar{i}} - s_i^+S_i^+) + \sum_{j\epsilon J} (t_{\bar{j}}T_{\bar{j}} - t_j^+T_j^+) - s^+S^+ + s^-S^- - t^+T^+ + t^-T^-$$

where the penalty (reward) is at the rate of $t^- (t^+)$ for the *total* deficit which include priorities. Similarly penalties (rewards) $s^-$, $s^+$, $t_j^+$, $s_{\bar{i}}$ and $s_i^+$ are defined.

Subject to the following constraints:

$$\text{(54)} \quad \sum_{j\epsilon J} e_{ij}x_{ij} = a_i + S_i^+ - S_i^- \qquad \text{for } i\epsilon I,$$

$$\text{(55)} \quad \sum_{i\epsilon I} x_{ij} = b_j + T_j^+ - T_{\bar{j}} \qquad \text{for } j\epsilon \mathscr{J},$$

$$\text{(56)} \quad \sum_{i\epsilon \mathscr{I}} S_i^+ = S^+;$$

$$\text{(57)} \quad \sum_{i\epsilon \mathscr{I}} S_i^- = S^-,$$

$$\text{(58)} \quad \sum_{i\epsilon \mathscr{I}} T_{\bar{j}} = T^+;$$

$$\text{(59)} \quad \sum_{j\epsilon \mathscr{J}} T_{\bar{j}} = T^-, \text{ and}$$

$$\text{(60)} \quad x_{ij}, S_i^+, S_{\bar{i}}, T_j^+ T_{\bar{j}} \geq 0 \qquad \text{for } i\epsilon I \text{ and } j\epsilon J.$$

(From (56)–(60), we have $S^+$, $S^-$, $T^+$, $T^- \geq 0$.)

This problem (53)–(60) can be converted to the generalized capacitated transportation problem due to the following.

Let

$$g_i^+ = g_i - s_i^+; \qquad g_{\bar{i}} = g_i + s_{\bar{i}}$$

$$h_j^+ = h_j - t_i^+; \qquad h_{\bar{j}} = h_j + t_{\bar{j}}$$

and

$$Z_0 = \sum_{i\epsilon I} g_i a_i + \sum_{j\epsilon J} h_j b_j$$

to get (53) as

$$\text{(61)}\qquad \text{Min } Z = Z_0 + \sum_{i\epsilon I}\sum_{j\epsilon J} c_{ij}x_{ij} + \sum_{i\epsilon I}(g_i^+S_i^+ + g_i^-S_i^-) + \sum_{j\epsilon J}(h_j^+T_j^+ + h_j^-T_j^-) - s^+S^+ + s^-S^- - t^+T^+ + t^-T^-.$$

Add $N$ to both sides of (54) and (55) to yield

$$\text{(62)}\qquad \sum_{j\epsilon J} e_{ij}x_{ij} + (N - S_i^+) + S_i^- = a_i + N \qquad \text{for } i\epsilon I,$$

$$\text{(63)}\qquad \sum_{i\epsilon I} x_{ij} + (T - T_j^+) + T_j^- = b_j + N \qquad \text{for } j\epsilon J.$$

$$\text{(64)}\qquad \sum_{i\epsilon \mathscr{I}} (N - S_i^+) + S^+ = mN \text{ and}$$

$$\text{(65)}\qquad \sum_{j\epsilon \mathscr{J}} (N - T_j^+) + T^+ = nN.$$

Further (57) and (59) can be viewed as

$$\text{(66)}\qquad \sum_{i\epsilon \mathscr{I}} S_i^- + (N - S^-) = N \text{ and}$$

$$\text{(67)}\qquad \sum_{j\epsilon \mathscr{J}} T_j^- + (N - T^-) = N.$$

Recalling $K_M$ from (11) and $K_P$ from (12) (the total volumes) we add to (11), (64) and (65) and subtract from it the *sum* of (62) over $i$, to get

$$\text{(68)}\qquad S^+ + (N - S^-) = K_M + N - \sum_{i\epsilon I} a_i.$$

Similarly we get for columns,

$$\text{(69)}\qquad T^+ + (N - T^-) = K_P + N - \sum_{j\epsilon \mathscr{J}} b_j$$

Let $Z_0' = Z_0 + \sum_{i\epsilon I} g_i^+N + \sum_{j\epsilon J} h_j^+N + s^-N + t^-N$; Then (60) becomes

$$\text{(70)}\qquad Z = Z_0' + \sum_{i\epsilon I}\sum_{j\epsilon J} c_{ij}x_{ij} - \sum_{i\epsilon I} g_i^+(N - S_i^+) + \sum_{i\epsilon I} g_i^-S_i^- - \sum_{j\epsilon J} h_j^+(N - T_j^+) + \sum_{j\epsilon J} h_j^-T_j^- - s^+S^+ - s^-(N - S^-) - t^+T^+ - t^-(N - T^-).$$

Since $S_i^+$, $S^-$, $T_j^+$, $T^-$ are defined to be $\geq 0$ we have

(71) $$(N - S_i^+) \leq N \quad \text{for } i \epsilon I, \qquad (N - T_j^+) \leq N \quad \text{for } j \epsilon J$$

$$(N - S^-) \leq N \quad \text{and} \quad (N - T^-) \leq N.$$

Thus (62) to (70) yield a $(m+4)$ row $(n+4)$ column capacitated generalized transportation problem as given in Figure 1. Again Algorithm 1 with very minor modifications can be used after solving with $K_M = K_a$ and $K_P = K_b$ and choosing options of relevant interest and applying proper cell rim operators $\delta R^+_{m+4,\,n+1}$ or $\delta R^+_{m+1,\,n+4}$ as the case may be.

| MCH. TYPES / PRODUCT TYPES | 1 2 ... j ... n | SLACK COLUMN n+1 | n+2 | n+3 | n+4 | MACHINE HOURS AVAILABLE | EQUATION # |
|---|---|---|---|---|---|---|---|
| 1<br>2<br>⋮<br>i<br>⋮<br>m | $\begin{bmatrix} x_{ij} \\ c_{ij} \\ e_{ij} \end{bmatrix}$ | $\begin{bmatrix} x_{i,n+1} \\ 0 \\ 1 \end{bmatrix}$ | $\begin{bmatrix} * \\ N-S_i^+ \\ -g_i^+ \\ 1 \end{bmatrix}$ | $\begin{bmatrix} S_i^- \\ g_i^- \\ 1 \end{bmatrix}$ | | $a_1+N$<br>.....<br>$a_i+N$<br>.....<br>.....<br>.....<br>$a_m+N$ | (62) |
| m+1 FICT. MCHNE. | $\begin{bmatrix} x_{m+1,\,j} \\ M \\ 1 \end{bmatrix}$ | $\begin{bmatrix} x_{m+1,\,n+1} \\ 0 \\ 1 \end{bmatrix}$ | | | | N+N | |
| m+2 | * $\begin{bmatrix} N-T_j \\ -h_j^+ \\ 1 \end{bmatrix}$ | | | | $\begin{bmatrix} T^+ \\ -t^+ \\ 1 \end{bmatrix}$ | nN | (65) |
| m+3 | $\begin{bmatrix} T_j^- \\ h_j^- \\ 1 \end{bmatrix}$ | | | | $\begin{bmatrix} * \\ N-T \\ -t \\ 1 \end{bmatrix}$ | N | (67) |
| m+4 | | | $\begin{bmatrix} S^+ \\ -s^+ \\ 1 \end{bmatrix}$ | $\begin{bmatrix} * \\ N-S \\ -s^- \\ 1 \end{bmatrix}$ | $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ | $K_M+N$<br>$-\Sigma a_i$ | (68) |
| DEMANDS | $b_1+N, \ldots, b_n+N$ | NO CONSTRAINT | mN | N | $K_P+N$<br>$-\Sigma b_j$ | | |
| EQUA. # | (63) | | (64) | (65) | (69) | | |

LEGEND $\begin{bmatrix} \text{VAR. NAME} \\ \text{UNIT COST} \\ \text{MULTIPLIER} \end{bmatrix}$

CROSS HATCHED BLOCKS DENOTE THAT CORRESPONDING $X_{ij}$ SHOULD BE MADE ZERO BY SETTING $c_{ij} = M$ (A LARGE PENALTY)

* DENOTES THAT THE UPPER BOUNDS FOR THESE $X_{ij}$ ARE N

FIGURE 1

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# A NOTE ON THE RELATIONSHIP BETWEEN SOME SPECIALIZED BINARY ALGORITHMS AND THE ALGORITHM OF LAND AND DOIG

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**ABSTRACT**

This note examines the relationship between several specialized binary algorithms and the pioneering work of Land and Doig. This relationship may provide new ideas which could enhance the effectiveness of these algorithms.

## I. INTRODUCTION

Several specialized algorithms have been developed for the binary problem:

$$\text{minimize } x_0 = CX, \tag{1}$$

subject to

$$AX + S = b \tag{2}$$

$$0 \leq x_j \leq 1, \qquad S \geq 0 \tag{3}$$

$$x_j = (0, 1), \qquad \text{all } j. \tag{4}$$

Among the more frequently cited specialized methods are those of Cabot and Hurter [2], Elmaghraby [3], Raghavachari [5], and Szwarc [6]. In all these methods, the solution space is converted into an equivalent continuous one by using various schemes for deleting the integer condition (4). The optimum (of the mixed or pure zero-one problem) is then recognized to occur at an extreme point of the continuous solution space. The indicated methods differ primarily in the details for identifying the optimum extreme point. For the purpose of comparison with the algorithm of Land and Doig [4], the specialized methods, taken in their chronological order of appearing in the literature, are discussed briefly.

Elmaghraby's and Szwarc's algorithms solve the mixed zero-one problem. Although the two methods were developed independently, they are based on similar ideas. The approach utilizes cuts which, starting from the continuous optimum, will rank the extreme points of the original solution space (according to their objective function value) until the extreme point satisfying the integrality condition is reached (for the first time). This is the optimum point. The cuts are defined parallel to the objective function, and each new cut is constructed to pass through the extreme point that is closest to the immediately preceding cut. "Closeness" here is measured in terms of the value of the objective function.

Cabot and Hurter consider the pure zero-one problem. Their modified continuous problem is

defined by dropping the integer restriction (4) and purturbing the right-hand side of constraints (2) in a manner that guarantees that the modified problem has the same integer feasible solutions as the original binary problem. They prove that a necessary and sufficient condition for attaining a feasible integer solution is that the slack variables associated with (purturbed) constraint (2) in the modified problem be basic and positive. This means that such a solution must be an extreme point of the modified problem, which also must correspond to an extreme point of the continuous space (2) and (3) in the original problem.

Cabot and Hurter do not provide an algorithm for finding solutions in which all the slacks of (2) are basic and positive.* However, under the assumption that this can be done, the corresponding value of the objective function, $x_0$, is used to define a bounding constraint of the form $CX < x_0$. After augmenting this constraint to the modified problem, a new solution is "found" in which all the slacks are basic and positive. The process of updating the bounding constraint is repeated until no further basic feasible solution can be determined. In this case the last feasible solution is the optimum.

Raghavachari modifies the objective function so that a very high penalty is incurred if any binary variable is assigned a fractional value. (The same idea of modifying the objective function and, in particular, the interpretation in terms of assigning a high penalty to fractional values was also conceived independently by Taha [7].) The resulting objective function becomes concave over the continuous solution space defined by constraints (2) and (3). In this case, the optimum of the modified problem continues to occur an extreme point of (2) and (3). The algorithm utilized to solve the concave problem is due to Tuy [9], and is based on developing linear cuts which will exclude all the extreme points having worse values of the objective function than that associated with a current *local* minimum. (A local minimum is necessarily an integer point in this case since fractional values of binary variables yield very high penalties.) The process is repeated by identifying a new local minimum of the *new* solution space. The algorithm terminates when no new local minima can be located. As in the Cabot-Hurter approach, Raghavachari does not give a specific procedure for locating local minima.

## II. RELATIONSHIP TO THE LAND-DOIG ALGORITHM

The algorithms of Elmaghraby, Szwarc, Cabot and Hurter, and Raghavachari are based on some type of (implicit or explicit) cut which is developed from information in the objective function. Specifically, Elmaghraby's and Szwarc's procedures "rank" the extreme points of the continuous solution space (starting from the continuous optimum) according to their value of the objective function. While Cabot and Hurter, and Raghavachari "rank" feasible integer extreme points by utilizing constraints or cuts which are based on the objective function.

The Land-Doig algorithm, when specialized to the zero-one case, can be shown to be based on the same general ideas utilized in the above algorithms. Specifically, (i) the Land-Doig algorithm recognizes that the optimum occurs at an extreme point of the continuous space (2) and (3), and (ii) it utilizes a ranking procedure of the extreme points, based on the objective function value. Perhaps the main reason for not generally recognizing these properties is that the Land-Doig algorithm is originally

*Interestingly, investigation of the earlier work of Balas [1] (see Taha [7]) shows that the *additive* algorithm is actually based on a weaker version of the Cabot and Hurter condition. Specifically, a *sufficient* condition for a feasible solution in Balas' algorithm is that the slacks in constraint (2) be basic and *nonnegative*, in which case the binary variables are nonbasic at zero or one level. Perhaps this result shows that solutions based on the Cabot-Hurter approach can be determined in a similar manner.

developed for the general integer problem, and therefore, its special properties for the zero-one case have not been given explicitly.

The above observations concerning the Land-Doig algorithm are verified by the following theorem:

THEOREM: Let the value of $x_0$ at the current extreme point be $x_0^*$ and let $Q$ be the set of *extreme points* defining the (bounded) solution space of the *associated* linear program. Assume that the value of the variable $x_j$ at this extreme point is fractional that is, $0 < x_j < 1$. Then, the next best extreme point *relative to the current extreme point* (node) that must be considered in applying the bounding rule corresponds to

$$x_0 = x_0^j = \min_{k=0,1} \left( \min_{X} \{CX | X \epsilon Q, x_j = k\} \right).$$

In this case, all the *extreme points*, $\bar{X}$, of (2) and (3) satisfying

$$\bar{X} = \{X \epsilon Q | x_j \neq k (k = 0 \text{ or } 1), \quad x_0^* < x_0(X) < x_0^j\}$$

have been *implicitly* accounted for, where $x_0(X)$ is the objective function value associated with $X$.

PROOF: The proof is summarized in three basic points:

(i) Since the value of $x_j$ must be zero or one, any extreme point that does not satisfy this condition must be discarded.

(ii) Because $x_j = 0$ and $x_j = 1$ are boundary planes of the solution space (2) and (3), *any* extreme point satisfying $x_j = 0$ or $x_j = 1$ must also be an extreme point of (2) and (3).

(iii) Since the finite optimum solution to the current linear program must occur at an extreme point, the extreme point in $Q$ having the *smallest* value of $x_0$ while satisfying the binary condition on $x_j$ must correspond to

$$\min_{k=0,1} \{ \min_{X} (CS | X \epsilon Q, x_j = k) \}.$$

This completes the proof.

To illustrate the above theorem, consider the graphical example in Figure 1. The extreme points of the solution space are ranked according to their corresponding values of $x_0$ with ① corresponding to the smallest and ⑧ to the largest.

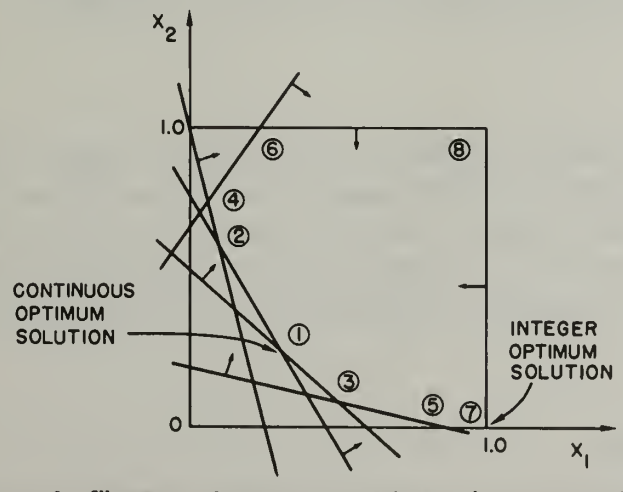


FIGURE 1. Illustration of extreme point ranking in the zero-one problem.

The Land-Doig algorithm starts from point ①. $x_1$ is selected for imposing condition (4) since it has a larger fractional value than $x_2$ (rule of thumb). Thus $\min_X \{CX \mid AX \leq b, x_1=1\}$ yields point ⑦ while $\min_X \{CX \mid AX \leq b, x_1=0\}$ yields no feasible solution, and ⑦ becomes the next ranked extreme point. But since it satisfies constraint (4), it is the optimal point. This actually means that the *nonpromising* extreme points ②, ③, ④, ⑤, and ⑥ have been *implicitly* ranked.

It must be noted that although the above theorem is proved for the *pure* zero-one case, it is equally applicable to the *mixed* zero-one case. This follows since the optimum solution must still occur at an extreme point of the solution space. In this case, the Land-Doig algorithm will be seeking the first *ranked* extreme point satisfying the binary-restricted variables.

## III. DISCUSSION

If Elmaghraby's and Szwarc's algorithms are utilized to solve the problem of Figure 1, the extreme points ①, ②, . . ., and ⑦ must be explicitly ranked before the optimum (at ⑦) is obtained. The Land-Doig algorithm is thus more "selective" since it ranks only those extreme points of the continuous space that are promising.

Because the Land-Doig algorithm identifies extreme point solutions, it is possible to utilize it to identify feasible integer solutions for use with Cabot and Hurter's and Raghavachari's approaches. Cabot and Hurter's and Raghavachari's approaches, however, when detailed for computational purposes must be accomplishing the same basic steps used in the Land-Doig method. Specifically, Cabot and Hurter's bounding constraint and Raghavachari's cut are equivalent to setting an upper bound on the value of the objective function (minimization problem). This is in essence what the Land-Doig algorithm does when an integer feasible point is located. Perhaps the main difference between the two methods is that the Land-Doig algorithm provides a well-defined procedure for locating integer feasible solutions, while Cabot and Hurter, and Raghavachari do not give such a procedure. Also, they express the upper bound explicitly in the form of a bounding constraint or a cut while the Land-Doig method utilizes the bound "implicitly" in the branch-and-bound procedure.

A recent paper by Zwart [10] provides a counter-example which shows that Tuy's algorithm may not converge. This conclusion makes Raghavachari's approach less promising.

## IV. CONCLUSION

This note provides a unifying framework for the specialized methods of Elmaghraby, Szwarc, Cabot and Hurter, and Raghavachari which suggests that they are all based on the same general idea; namely, that the optimum solution occurs at an extreme point of the continuous solution space. Moreover, these methods are shown to be closely related to the pioneering work of Land and Doig. Perhaps these relationships can assist in shedding light on new ideas for tackling the integer zero-one problem.

## V. ACKNOWLEDGMENT

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# A NOTE ON A SCHEDULING PROBLEM WITH DUAL CRITERIA

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Consider the $n$ job, one machine scheduling problem with the following dual criteria. Let job $j$ $(j=1(1)n)$ have process time $p_j$ and have arbitrary nondecreasing penalty function $c_j(t)$ denoting the cost of completing the job at a time $t$. For example, the weighted tardiness criterion gives $c_j(t)=v_jT_j$, where $T_j=\max\ (0,\ t-d_j)$ is the tardiness of job $j$ with due date $d_j$, and $v_j$ is its value. All jobs are available at time zero, and their completion or flow time under schedule $S$ is denoted $F_j(S)$, or simply $F_j$. The primary criterion is to minimize the maximum penalty, which we denote

$$c_{\max}=\max_j c_j(F_j).$$

Within the subset of schedules which achieves this, we seek the schedule with minimal total (or mean) flow time, $F=\Sigma_j F_j$.

A simple algorithm to solve the problem with primary criterion only has been presented by Lawler [3]. Using the notation $n/1/C$ to denote the $n$ job, one machine scheduling problem with criterion $C$, we have:

THEOREM 1 (Lawler): For $n/1/c_{\max}$, there exists an optimal schedule in which job $k$ is last, where

$$c_k(p)=\min_j c_j(p), \text{ and}$$

where $p=\Sigma_j p_j$ is the makespan.

Thus, we can iteratively place that job last which is least costly there.

Heck and Roberts [1] have extended an algorithm of Smith [4] to produce the following theorem for the dual criterion of minimizing total flow time, given that the maximum tardiness is minimal.

THEOREM 2 (Heck & Roberts): For $n/1/F\mid \min T_{\max}$, find the minimal $T_{\max}$, $T_m\equiv \min_S \max_j T_j(S)$, which is achieved by scheduling in order of nondecreasing due dates [2]. Let $L$ denote the subset of jobs which may be performed last without violating the primary objective:

$$L=\{j: d_j\geq p-T_m\}.$$

Then there exists an optimal schedule for the dual criterion in which job $k$ is last, where

$$p_k=\max_{j\in L} p_j.$$

We now extend this theorem to incorporate the more general conditions of Lawler.

THEOREM 3: For $n/1/F\mid \min\ c_{\max}$, find the minimal $c_{\max}$, $c_m\equiv \min_S \max_j c_j[F_j(S)]$, by using Lawler's theorem. Let $L$ denote the subset of jobs which may be performed last without violating the primary objective:

$$L=\{j: c_j(p)\leq c_m\}.$$

Then there exists an optimal schedule in which job $k$ is last, where

$$p_k = \max_{j \in L} p_j.$$

PROOF: Start with any schedule, $S$, which achieves the primary objective of minimizing $c_{\max}$, but which has job $n \neq k$ last, with $p_n < p_k$. We assume jobs are indexed in the order they appear in $S$. Consider interchanging $k$ and $n$ to produce schedule $S'$.

(a) Except for job $k$, all jobs are completed earlier in $S'$ than in $S$, and since $c_j(\cdot)$ are nondecreasing functions of completion time, their costs can only diminish. Thus, since $k \in L$, the primary criterion remains satisfied.

(b) The change in the secondary objective is

$$F(S) - F(S') = (n-k)\ (p_k - p_n) > 0.$$

Thus, any schedule, $S$, violating the requirements of the theorem can be improved. Q.E.D.

Finally, it should be pointed out that the extension of this and earlier results to the case of total *weighted* flow times, $F_w = \Sigma_i w_i F_i$, as the secondary criterion does not go through as simply as has been stated by others. Thus, Smith [4] states without proof the following.

CONJECTURE: For $n/1/F_w|T_{\max} = 0$, let $L = \{j : d_j \geq p\}$. Then there exists an optimal schedule in which job $k$ is last, where

$$p_k/w_k = \max_{i \in L} p_i/w_i.$$

Though this seems to be a very plausible combination of two valid theorems, it is false, as the following 3-job counter-example shows.

| $i$ | 1 | 2 | 3 |
|---|---|---|---|
| $d_i$ | 35 | 26 | 35 |
| $p_i$ | 6 | 20 | 4 |
| $w_i$ | 5 | 1 | 4 |

Smith's algorithm gives $S = (3, 2, 1)$, with $F_w = 190$, while the true optimum is $S' = (1, 2, 3)$ with $F_w = 176$. Heck and Roberts [1] cite this result to justify extending their theorem to include weights.

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